Chapter 5. Initial Value Problems for Ordinary Differential Equations

5.1 The Elementary Theory of Initial-Value Problems

Let $y(t)$ be a function of t defined for $a \le t \le b$. An equation that involves t, $y(t)$ and some of its derivatives $y', y'', \dots$ is called ordinary differentia equation (ODE).

Ex: 1) $y'' = 2$ ← second order ODE
2) $y' = y\cos t$ ← first order ODE
3) $y'' = 2y' + 3y + 5$ ← second order ODE

The highest derivative that participates gives the order of the ODE.

We will consider primarily first order ODE

A solution of the ODE is a function $y(t)$ which has the necessary derivative and substituted in the equation turns it into identity.

Ex: For the equation

$$y' = y\cos t \quad 0 \le t \le 1$$

the function

$$y(t) = e^{\sin t}$$

is a solution.

Check: $y'(t) = e^{\sin t} \cdot \cos t = y\cos t$

However, an ODE may have more than one solution

$$y(t) = 2e^{\sin t}$$

is also a solution

$$y'(t) = 2e^{\sin t} \cos t = y\cos t$$

To specify a <u>unique</u> solution we have to give an additional condition — typically that is the value of the solution for the initial value of t

$$\begin{cases} y'(t) = y\cos t \\ y(0) = 1 \end{cases}$$

This problem has a unique solution $y(t) = e^{\sin t}$. This type of problems are called first order initial-value problems.

In this chapter we consider the problem

(1) $$y' = f(t,y) \qquad a \le t \le b$$

subject to the initial condition

(2) $$y(a) = \alpha$$

The problem (1), (2) is called first order initial-value problem.

Def A differentiable function $y(t)$ which satisfies (1) and (2) is called a solution.

Not all differential equations are easy to solve. In this chapter we study methods to find an approximation of the solution.

When solving numerically an ODE it is important to know that the initial-value problem has a unique solution.

It turns out that that is the case if $f(t,y)$ satisfies the so-called Lipschitz condition.

Def. A function $f(t,y)$ is said to satisfy a <u>Lipschitz condition</u> in the variable y on a set D in the plane if a constant $L>0$ exists such that

$$|f(t,y_1)-f(t,y_2)| \le L|y_1-y_2|$$

whenever (t,y_1), (t,y_2) are in D. The constant L is called <u>Lipschitz constant</u> for f.

Ex. Show that the function

$$f(t,y)=\frac{4t^3y}{1+t^4} \qquad 0\le t\le 1$$

satisfies a <u>Lipschitz condition</u> and find the Lipschitz constant

<u>Solution:</u>

$$\left|\frac{4t^3y_1}{1+t^4}-\frac{4t^3y_2}{1+t^4}\right| = \left|\frac{4t^3}{1+t^4}\right||y_1-y_2| \le 4\cdot\frac{1}{2}|y_1-y_2|$$

increasing in $[0,1]$ $\quad = 2|y_1-y_2|$

Thus, $L=2$.

Ex. Show that the function

$$f(t,y) = t\cos y \qquad 1 \le t \le 2$$

satisfies Lipschitz condition and find the Lipschitz constant

Solution:

$$|t\cos y_1 - t\cos y_2| = t|\cos y_1 - \cos y_2|$$

Consider $\cos y_2 = \cos(y_1 + (y_2 - y_1))$ and expand in Taylor's series

$$\cos(y_1 + (y_2 - y_1)) = \cos y_1 + \frac{d}{dy}\cos y \Big|_{y=y_1} (y_2 - y_1) + h.o.t$$

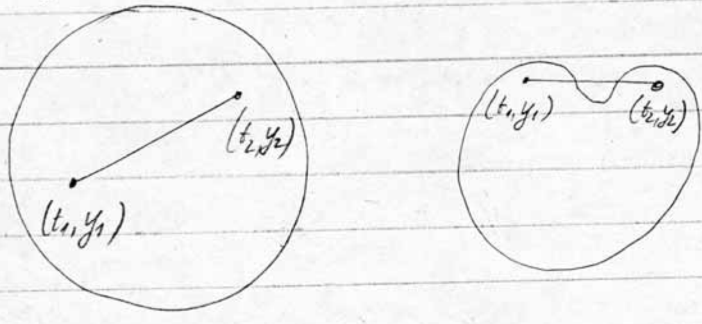
$$t|\cos y_1 - \cos y_2| = t\,|\cancel{\cos y_2} - \cancel{\cos y_1} + \sin y_2 (y_2 - y_1)|$$

$$= t|\sin y_1||y_2 - y_1| \le 2|y_2 - y_2|$$

$\Rightarrow L = 2.$

There is a simple criterion that gives Lipschitz condition for a function f on a set D in the plane.

Def. A set D in the plane is called <u>convex</u> if whenever it contains the points (t_1, y_1) and (t_2, y_2) it also contains the segment that connects them.



convex   not convex

Theorem 5.3. 1) Let D - convex
2) For $f(t,y)$ defined on D - let a constant $L > 0$ exists with

$$\left| \frac{\partial f}{\partial y}(t,y) \right| < L \quad \text{for all } (t,y) \text{ in } D.$$

Then f satisfies a Lipschitz condition on D in the variable y with Lipschitz constant L.

Ex: Show that the function

$$f(t,y) = \cos(t+y) \qquad 1 \le t \le 2$$

satisfies a Lipschitz condition and determine the Lipschitz constant

$D = \{(t, y) : 1 \le t \le 2 \ , \ -\infty < y < \infty\}$ - convex

$$\frac{\partial f}{\partial y} = -\sin(t+y)$$

$|\sin(t+y)| \le 1$ for all (t, y) in D.

Thus, $f(t, y)$ satisfies Lipschitz condition with

$L = 1$.