MAC 2313-02 Calculus III
Final Exam
12/13/2012 Print Name ____________________________

1. (10pt) The vector equation of a curve C is $\vec{r}(t) = \langle 3\cos t, 3\sin t, 4t\rangle$.

(a) Find the arc length between the points $P(3, 0, 0)$ and $Q(3, 0, 8\pi)$ on the curve C.
$\vec{r}\,' = \langle -3\sin t, 3\cos t, 4\rangle$.
The point P corresponds to the parameter value $t = 0$, while Q corresponds to $t = 2\pi$. Arc length between P and Q is given be

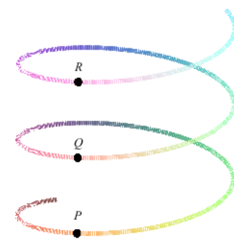
$$\int_0^{2\pi} |\vec{r}\,'(t)|\,dt = \int_0^{2\pi} \sqrt{(-3\sin t)^2 + (3\cos t)^2 + 4^2}\,dt = \int_0^{2\pi} 5\,dt = 10\pi.$$

(b) Find the coordinates of a point R other than $P(3, 0, 0)$ on the curve C, such that the arc length between R and Q is the same as the arc length between P and Q.

The arc length formula starts at Q is given by

$$\int_{2\pi}^{t} |\vec{r}\,'(u)|\,du = \int_{2\pi}^{t} \sqrt{(-3\sin u)^2 + (3\cos u)^2 + 4^2}\,du = \int_{2\pi}^{t} 5\,du$$

Setting $5(t - 2\pi) = 10\pi$ to see $t = 4\pi$, which corresponds to the point $R(3, 0, 16\pi)$.



2. (10pt) $f(x, y, z) = xe^y + ye^z + ze^x$.

(a) (3pt) Find the gradient of $f(x, y, z)$.
Solution:

$$\nabla f = \langle f_x, f_y, f_z\rangle = \langle e^y + ze^x,\ xe^y + e^z,\ ye^z + e^x\rangle.$$

$\square$

(b) (4pt) Find the directional derivative of f at the point $(0, 0, 0)$ in the direction of $\langle 0, 2, 1\rangle$.
Solution:

$$\begin{aligned}
\boldsymbol{u} &= \frac{\langle 0,2,1\rangle}{\sqrt{0^2+2^2+1^2}} = \frac{1}{\sqrt{5}}\langle 0,2,1\rangle \\
(\nabla f)(0,0,0) &= \langle 1,1,1\rangle \\
D_{\boldsymbol{u}} f(0,0,0) &= (\nabla f)(0,0,0)\cdot \boldsymbol{u} \\
&= \frac{1}{\sqrt{5}}\langle 1,1,1\rangle \cdot \langle 0,2,1\rangle \\
&= \frac{3}{\sqrt{5}}
\end{aligned}$$

$\square$

(c) (3pt) Find the maximum rate of change of f at the point $(0, 0, 0)$. In which direction does it occur?
Solution: The maximum rate of change occurs in the direction of the gradient

$$\nabla f(0, 0, 0) = \langle 1, 1, 1\rangle,$$

and the rate of change is given by the magnitude of the gradient

$$|\nabla f(0, 0, 0)| = \sqrt{3}.$$

$\square$

3. (10pt) Find the local maximum and minimum values and saddle points of $f(x, y) = x^2 - xy + y^2 + 9x - 6y + 10$, if they exist.

Solution:

$$\begin{aligned} f_x &= 2x - y + 9 \\ f_y &= -x + 2y - 6 \end{aligned}$$

Setting $f_x = 0, f_y = 0$ to get $x = -4, y = 1$. Thus, $(-4, 1)$ is the only critical point.

$$\begin{aligned} f_{xx} &= 2 \\ f_{yy} &= 2 \\ f_{xy} &= -1 \end{aligned}$$

Since $f_{xx} f_{yy} - f_{xy}^2 > 0$ and $f_{xx} > 0$ at the point $(-4, 1)$, $f(x, y)$ has a local minimum $f(-4, 1) = -11$ at the point $(-4, 1)$. $\square$

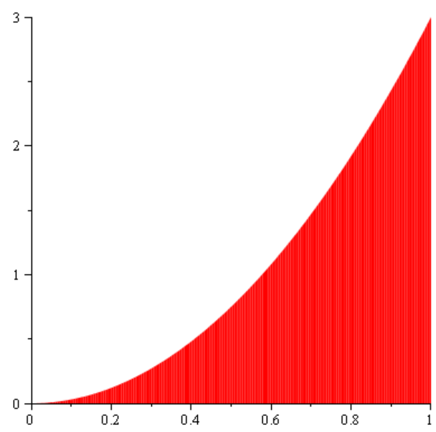
4. (10pt) Sketch the region of integral and calculate the iterated integral by first reversing the order of integration.

$$\int_0^3 \int_{\sqrt{y/3}}^1 e^{x^3}\, dx\, dy$$

Solution:

$$\begin{aligned} \int_0^3 \int_{\sqrt{y/3}}^1 e^{x^3}\, dx\, dy &= \int_0^1 \int_0^{3x^2} e^{x^3}\, dy\, dx \\ &= \int_0^1 3x^2 e^{x^3}\, dx \qquad \text{substitute } u = x^3 \\ &= \int_0^1 e^u\, du \\ &= e - 1 \end{aligned}$$

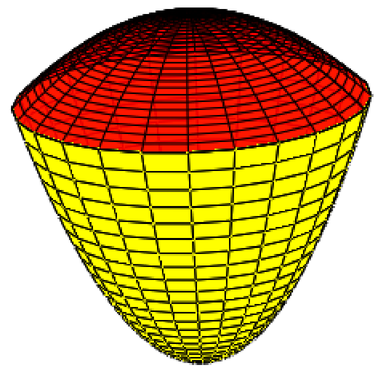
$\square$



5. (10pt) Find the volume of the solid bounded by the two paraboloids $z = 3x^2 + 3y^2$ and $z = 4 - x^2 - y^2$.
Solution: Setting $3x^2 + 3y^2 = 4 - x^2 - y^2$ to see $x^2 + y^2 = 1$, which is the intersection of the two paraboloids. So, the projection of the volume (i.e. D) on the xy-plane is the unit disk.

The volume is given by the double integral

$$\begin{aligned} & \iint_D \left[(4 - x^2 - y^2) - (3x^2 + 3y^2)\right] dA \\ = & \iint_D (4 - 4x^2 - 4y^2)\, dA \\ = & \int_0^{2\pi} \int_0^1 (4 - 4r^2) r \, dr \, d\theta \\ = & \int_0^{2\pi} \int_0^1 (4r - 4r^3)\, dr\, d\theta \\ = & \int_0^{2\pi} (2r^2 - r^4)\Big|_0^1 \, d\theta \\ = & \ 2\pi. \end{aligned}$$



6. (10pt) Find the volume of the solid that lies within the sphere $x^2 + y^2 + z^2 = 4$, above the xy-plane and below the cone $z = \sqrt{x^2 + y^2}$.
Solution: In spherical coordinates,

$$\begin{aligned} x &= \rho \sin\phi \cos\theta \\ y &= \rho \sin\phi \sin\theta \\ z &= \rho \cos\phi. \end{aligned}$$

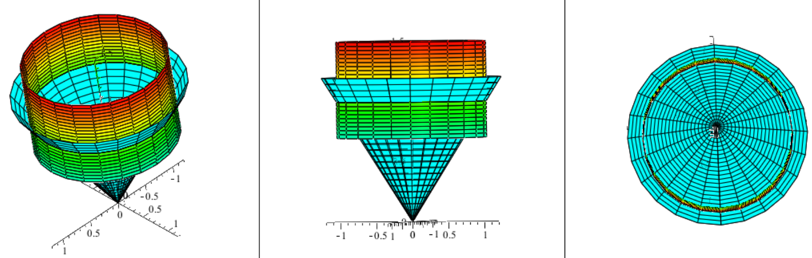
Plug them into $x^2 + y^2 + z^2 = 4$ to see the sphere is $\rho = 2$.
Plug them into $z = \sqrt{x^2 + y^2}$ to see the cone is $\phi = \frac{\pi}{4}$. Therefore,

$$\begin{aligned} V &= \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \\ &= \int_0^{2\pi} d\theta \int_{\pi/4}^{\pi/2} \sin\phi \, d\phi \int_0^2 \rho^2 \, d\rho \\ &= (2\pi) \left(\frac{\sqrt{2}}{2} \right) \frac{8}{3} \\ &= \frac{8}{3} \sqrt{2} \pi. \end{aligned}$$

$\square$

7. (10pt) Use Stokes' theorem to evaluate $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = x^2 z^2 \mathbf{i} + y^2 z^2 \mathbf{j} + xyz\mathbf{k}$ and S is the part of the paraboloid $z = x^2 + y^2$ that lies inside the cylinder $x^2 + y^2 = 1$, oriented upward.

(Hint: Stokes' theorem: $\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$. The boundary of S (i.e. the curve C) is the intersection of the paraboloid with the the cylinder.)



Solution:

The boundary of S is the intersection of the paraboloid and the cylinder, which is a circle with radius 1 on the plane $z = 1$. Thus, the boundary C has a parametric equation

$$\begin{aligned} \boldsymbol{r}(\theta) &= \langle \cos\theta,\ \sin\theta,\ 1\rangle, \qquad 0 \le \theta \le 2\pi, \\ \text{and} \qquad \boldsymbol{r}'(\theta) &= \langle -\sin\theta,\ \cos\theta,\ 0\rangle. \end{aligned}$$

By Stokes' Theorem,

$$\begin{aligned} \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} &= \int_C \mathbf{F} \cdot d\mathbf{r} \\ &= \int_0^{2\pi} \mathbf{F} \cdot \boldsymbol{r}'(\theta)\, d\theta \\ &= \int_0^{2\pi} \langle (\cos\theta)^2(1)^2,\ (\sin\theta)^2(1)^2,\ (\cos\theta)(\sin\theta)(1)\rangle \cdot \boldsymbol{r}'(\theta)\, d\theta \\ &= \int_0^{2\pi} (-\cos^2\theta \sin\theta + \sin^2\theta\cos\theta)\, d\theta \\ &= \left.\frac{\cos^3\theta}{3}\right|_0^{2\pi} + \left.\frac{\sin^3\theta}{3}\right|_0^{2\pi} \\ &= 0 + 0 \\ &= 0. \end{aligned}$$

□

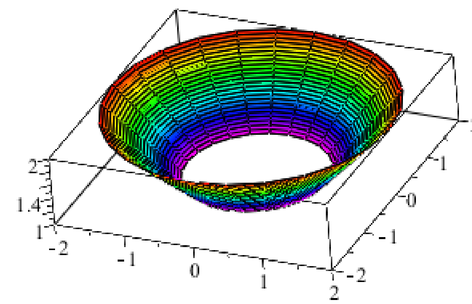
8. (10pt) Evaluate $\iint_S \mathbf{F}\cdot d\mathbf{S}$, where $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + \mathbf{k}$ and S is the part of the cone $z^2 = x^2 + y^2$ that lies between the planes $z = 1$ and $z = 2$, oriented upward.

Solution:

The parametric equation of S is $\boldsymbol{r}(a,\theta) = \langle a\cos\theta,\ a\sin\theta,\ a\rangle$, where $1 \le a \le 2$ and $0 \le \theta \le 2\pi$.(Note: the z component is a, because $z^2 = x^2+y^2 = (a\cos\theta)^2 + (a\sin\theta)^2 = a^2$. Then $z = a$.)

$$\begin{aligned} \boldsymbol{r}_a &= \langle \cos\theta,\ \sin\theta,\ 1\rangle \\ \boldsymbol{r}_\theta &= \langle -a\sin\theta,\ a\cos\theta, 0\rangle \\ \boldsymbol{r}_a\times\boldsymbol{r}_\theta &= \langle -a\cos\theta,\ -a\sin\theta,\ a\rangle \quad \text{(upward orientation)} \end{aligned}$$

$$\begin{aligned} &\iint_S \mathbf{F}\cdot d\mathbf{S} \\ =\ & \iint_D \mathbf{F}\cdot(\boldsymbol{r}_a\times\boldsymbol{r}_\theta)\,dA \quad D = [1,2]\times[0,2\pi] \\ =\ & \int_0^{2\pi}\int_1^2 \langle a\cos\theta, a\sin\theta, 1\rangle\cdot\langle -a\cos\theta,\ -a\sin\theta,\ a\rangle\,da\,d\theta \\ =\ & \int_0^{2\pi}\int_1^2 (-a^2 + a)\,da\,d\theta \\ =\ & 2\pi\left(-\frac{a^3}{3} + \frac{a^2}{2}\right)\Big|_1^2 \\ =\ & -\frac{5\pi}{3} \end{aligned}$$



$\square$

Alternatively, S can be parametrized as

$$\boldsymbol{r}(x,y) = \langle x,\ y,\ \sqrt{x^2+y^2}\rangle,$$

where x, y belongs to the region E between two circles $x^2+y^2 = 1^2$ and $x^2+y^2 = 2^2$.

$$\begin{aligned} \boldsymbol{r}_x &= \langle 1,\ 0,\ \frac{x}{\sqrt{x^2+y^2}}\rangle \\ \boldsymbol{r}_y &= \langle 0,\ 1, \frac{y}{\sqrt{x^2+y^2}}\rangle \\ \boldsymbol{r}_x\times\boldsymbol{r}_y &= \langle -\frac{x}{\sqrt{x^2+y^2}},\ -\frac{y}{\sqrt{x^2+y^2}},\ 1\rangle \quad \text{(upward orientation)} \\ \iint_S \mathbf{F}\cdot d\mathbf{S} &= \iint_E \mathbf{F}\cdot(\boldsymbol{r}_x\times\boldsymbol{r}_y)\,dA \\ &= \iint_E \langle x,\ y,\ 1\rangle\cdot\langle -\frac{x}{\sqrt{x^2+y^2}},\ -\frac{y}{\sqrt{x^2+y^2}},\ 1\rangle\,dx\,dy \\ &= \iint_E (-\sqrt{x^2+y^2} + 1)\,dx\,dy \qquad \text{use polar coordinates} \\ &= \int_0^{2\pi}\int_1^2 (-r+1)r\,dr\,d\theta \\ &= 2\pi\left(-\frac{r^3}{3} + \frac{r^2}{2}\right)\Big|_1^2 \\ &= -\frac{5\pi}{3}. \end{aligned}$$

9. (10pt) Find the area of part of the surface $3x + 4y + z = 6$ that lies in the first octant.
Solution:
One of the parametrizations about the surface plane S is

$$\mathbf{r}(x, y) = \langle x, y, 6 - 3x - 4y\rangle,$$

where $0 \le y \le \dfrac{6-3x}{4}, 0 \le x \le 2$. Then,

$$\begin{aligned} \mathbf{r}_x &= \langle 1, 0, -3\rangle, \\ \mathbf{r}_y &= \langle 0, 1, -4\rangle, \\ \mathbf{r}_x \times \mathbf{r}_y &= \langle 3, 4, 1\rangle \\ |\mathbf{r}_x \times \mathbf{r}_y| &= \sqrt{26} \end{aligned}$$

The area is given by

$$\begin{aligned} \iint_S dS &= \iint_S |\mathbf{r}_x \times \mathbf{r}_y|\, dy\, dx \\ &= \int_0^2 \int_0^{\frac{6-3x}{4}} \sqrt{26}\, dy\, dx \\ &= \sqrt{26} \int_0^2 \frac{6-3x}{4}\, dx \\ &= \sqrt{26}\left(\frac{6}{4}x - \frac{3}{8}x^2\right)\Big|_0^2 \\ &= \frac{3}{2}\sqrt{26}. \end{aligned}$$

10. (10pt)$\mathbf{F}(x, y) = x^2\mathbf{i} + y^2\mathbf{j}$.
(a) Show that $\mathbf{F}$ is conservative and find f such that $\nabla f(x, y) = \mathbf{F}(x, y)$.
Solution:
Set $P = x^2, Q = y^2$. Then $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$. Since $\mathbf{F}$ is well defined everywhere, $\mathbf{F}$ is conservative.

If $\nabla f(x, y) = \mathbf{F}(x, y)$, then

$$\begin{aligned} f_x &= x^2 \\ f_y &= y^2 \end{aligned}$$

By partial integration,

$$f = \frac{1}{3}x^3 + \frac{1}{3}y^3 + K \qquad \text{where } K \text{ is a constant.}$$

□

(b) Use the result in part (a) and the Fundamental Theorem for line integral to calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the arc of the parabola $y = 2x^2$ from $(0, 0)$ to $(1, 2)$.
Solution:

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= f(1, 2) - f(0, 0) \\ &= 3. \end{aligned}$$

□