6.4 The Determinant of a Matrix

The determinant of a square matrix is a number associated with the matrix. We denote it by

$$|A| \quad \text{or} \quad \det(A)$$

If A is an $n \times n$ matrix

Def: The <u>minor</u> M_{ij} is the determinant of the $(n-1)\times(n-1)$ submatrix of A obtained by deleting the ith row and jth column of the matrix A.

Ex: $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \qquad M_{23} = \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix}$

Def: The <u>cofactor</u> A_{ij} associated with M_{ij} is defined as

$$A_{ij} = (-1)^{i+j} M_{ij}$$

Ex: $A_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} = (-1) \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix}$

Now we define the determinant inductively.

Def: If $A=(a)$ then $\det(A)=a$

The determinant of $n\times n$ matrix A is given by

$$\det(A)=\sum_{j=1}^{n} a_{ij}A_{ij} \qquad i=1,2,\dots,n$$

(expansion by the elements of the i^{th} row) or by

$$\det(A)=\sum_{i=1}^{n} a_{ij}A_{ij} \qquad j=1,2,\dots,n$$

(expansion by the elements of the j^{th} column).

Ex. Compute the determinant of the matrix

a) $$A=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$$

$$\begin{vmatrix}1 & 2\\ 3 & 4\end{vmatrix} = 1\cdot M_{11}+(-1)^{1+2}\cdot 2M_{12} =$$
$$= 1\cdot 4-2\cdot 3 = 4-6=-2$$

b) $$A=\begin{pmatrix}1 & 2 & 0\\ 2 & 1 & -1\\ 3 & 1 & 1\end{pmatrix}$$

$$\begin{vmatrix} 1 & 2 & 0 \\ 2 & 1 & -1 \\ 3 & 1 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} - 2 \begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} + 0 \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix}$$

$$= 1 \cdot (1+1) - 2(2+3) = 2 - 2 \cdot 5 = -8$$

Theorem 6.15: If A is $n \times n$ matrix

(a) If any row or column of A consists of zeroes, then $\det(A) = 0$

(b) If A has 2 rows or 2 columns that are the same then $\det(A) = 0$

(c) If $\tilde{A}$ is obtained by the operation $(E_i) \leftrightarrow (E_j)$ then

$$\det(\tilde{A}) = -\det(A)$$

(d) If $\tilde{A}$ is obtained from A by the operation $(\lambda E_i) \to (E_i)$ then

$$\det(\tilde{A}) = \lambda \det(A)$$

(e) If $\tilde{A}$ is obtained from A by the operation

$$(\lambda E_i + E_j) \to (E_j)$$

then $\det(\tilde{A}) = \det(A)$

(f) If B is $n \times n$ matrix

$$\det(AB) = \det A \cdot \det B$$

(g) $\det(A^t) = \det A$

(h) If A^{-1} - exists

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

i. If A is upper (lower) triangular or diagonal det(A) is the product of elements along the diagonal.

Ex. Use Theorem 6.15 to compute the determinant

$$\begin{vmatrix} 2 & 0 & 1 & 2 \\ 1 & 1 & 0 & 2 \\ 2 & -1 & 3 & 1 \\ 3 & -1 & 4 & 3 \end{vmatrix} =$$

$$= \begin{vmatrix} 2 & 0 & 1 & 2 \\ 1 & 1 & 0 & 2 \\ 3 & 0 & 3 & 3 \\ 4 & 0 & 4 & 5 \end{vmatrix} =$$

$$1 \cdot (-1)^{2+2} \begin{vmatrix} 2 & 1 & 2 \\ 3 & 3 & 3 \\ 4 & 4 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 3 & 0 \\ 0 & 4 & 1 \end{vmatrix} = \begin{vmatrix} 3 & 0 \\ 4 & 1 \end{vmatrix}$$

$$= 3$$

Theorem 6.16 The following statements are equivalent

(a) The system $A\vec{x} = \vec{0}$ has a unique sol. $\vec{x} = \vec{0}$

(b) The system $A\vec{x} = \vec{b}$ has a unique solution for any $\vec{b}$

(c) The matrix A is nonsingular; that is A^{-1} exis[ts]
(d) $\det A \neq 0$
(e) Gaussian elimination with row interchanges can be performed on the system $A\vec{x} = \vec{b}$ for any $\vec{b}$

Ex: Find all values of α so that the system

$$\begin{aligned} 2x_1 - x_2 + 3x_3 &= 5 \\ 4x_1 + 2x_2 + 2x_3 &= 6 \\ -2x_1 + \alpha x_2 + 3x_3 &= 4 \end{aligned}$$

has no solutions

$$\left(\begin{array}{ccc|c} 2 & -1 & 3 & 5 \\ 4 & 2 & 2 & 6 \\ -2 & \alpha & 3 & 4 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & -1 & 3 & 5 \\ 2 & 1 & 1 & 3 \\ -2 & \alpha & 3 & 4 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 2 & -1 & 3 & 5 \\ 0 & 2 & -2 & -2 \\ 0 & \alpha-1 & 6 & 9 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & -1 & 3 & 5 \\ 0 & 1 & -1 & -1 \\ 0 & \alpha-1 & 6 & 9 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 2 & -1 & 3 & 5 \\ 0 & \alpha-1 & -(\alpha-1) & -(\alpha-1) \\ 0 & \alpha-1 & 6 & 9 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & -1 & 3 & 5 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & \alpha+5 & \alpha+8 \end{array}\right)$$

$\alpha = -5$ makes the last eqn $0 = 3$
no solution.

6.5 Matrix Factorization

1) LU factorization for A where Gauss elimination with no row interch.

In this section we will see that Gauss elimination for solving

$$A\vec{x} = \vec{b}$$

can be used to factor the matrix A.

The factorization

$$A = LU$$

L - lower triangular; U - upper triangular

is particularly useful.

Assume $A = LU$. Then we can solve the system

$$LU\vec{x} = \vec{b}$$

Set $y = U\vec{x}$. First we solve

$$L\vec{y} = \vec{b}$$

then we solve

$$U\vec{x} = \vec{y}$$

This requires fewer computations provided we have $A = LU$.

Ex. Solve the linear system

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

First we solve the system

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

$\boxed{y_1 = 2}$ $\quad 2y_1 + y_2 = -1 \Rightarrow \boxed{y_2 = -5}$

$-y_1 + y_3 = 1$

$\boxed{y_3 = 3}$

Then we solve the system

$$U\vec{x} = \vec{y}$$

$$\begin{pmatrix} 2 & 3 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix}$$

$3x_3 = 3 \Rightarrow \boxed{x_3 = 1}$ $\quad -2x_2 + x_3 = -5$

$-2x_2 = -6$ $\quad \boxed{x_2 = 3}$

$2x_1 + 3x_2 - x_3 = 2$

$2x_1 + 9 - 1 = 2$

$2x_1 + 8 = 2$

$2x_1 = -6 \Rightarrow \boxed{x_1 = -3}$

Theorem 6.17. If Gauss elimination can be performed on $A\vec{x}=\vec{b}$ without row interchanges, then the matrix A can be factored into the product

$$A = LU$$

where L is the lower-triangular matrix

$$L = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ m_{21} & 1 & \cdots & 0 \\ \vdots & & & \\ m_{n1} & m_{n2} & \cdots & 1 \end{pmatrix}$$

where

$$m_{ji} = \frac{a_{ji}^{(i)}}{a_{ii}^{(i)}}$$

are the multipliers multiplying the pivot row and

$$U = \begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & \cdots & a_{2n}^{(2)} \\ & & \ddots & \\ & & & a_{nn}^{(n)} \end{pmatrix}$$

is the upper-triangular matrix which remains after the Gauss-elimination.

Ex. Consider the system

$$\begin{aligned} 2x_1 - x_2 + x_3 &= -1 \\ 3x_1 + 3x_2 + 9x_3 &= 0 \\ 3x_1 + 3x_2 + 5x_3 &= 4 \end{aligned}$$

Factor the matrix $A = LU$ ~~and then use the factorization to solve the system.~~

First we perform Gauss elimination

$$\left(\begin{array}{ccc|c} 2 & -1 & 1 & -1 \\ 3 & 3 & 9 & 0 \\ 3 & 3 & 5 & 4 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 2 & -1 & 1 & -1 \\ 0 & \frac{9}{2} & \frac{15}{2} & \frac{3}{2} \\ 0 & \frac{9}{2} & \frac{7}{2} & \frac{11}{2} \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 2 & -1 & 1 & -1 \\ 0 & \frac{9}{2} & \frac{15}{2} & \frac{3}{2} \\ 0 & 0 & -4 & 4 \end{array}\right)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ \frac{3}{2} & 1 & 1 \end{pmatrix} \qquad U = \begin{pmatrix} 2 & -1 & 1 \\ 0 & \frac{9}{2} & \frac{15}{2} \\ 0 & 0 & -4 \end{pmatrix}$$

Thus,

$$A = LU$$

Check:

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ \frac{3}{2} & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 1 \\ 0 & \frac{9}{2} & \frac{15}{2} \\ 0 & 0 & -4 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 1 \\ 3 & 3 & 9 \\ 3 & 3 & 5 \end{pmatrix}$$

The factorization discussed so far is called Doolittle's method and requires 1's to be on the diagonal of L. Crout's method requires 1's to be on the diagonal of U. Choleski's method requires $l_{ii} = u_{ii}$.

2) LU factorization for A where Gauss elimination needs row interchanges.

The operation "interchange of rows" can be performed by multiplying A by a special type of matrix, called permutation matrix.

An $n \times n$ permutation matrix P is obtained by rearranging the rows of I_n.

Ex: $P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ permutation matrix

Notice that P is obtained from I_n by intergaging the first and second row of I_n. Multiplying any matrix A by P on the left has the effect of interchanging row one and two.

Thus, the matrix PA can be factored into

$$PA = LU$$

L - lower triangular, U - upper triangular

$$P^{-1}PA = P^{-1}LU$$
$$A = P^{-1}LU$$

Since $P^{-1} = P^t$

$$A = P^t LU = (P^t L)U$$

Thus, the matrix U is still upper-triangular but $P^t L$ is not lower triangular.

Ex.
$$A = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 1 & 1 & 4 & 3 \\ 2 & -1 & 2 & 4 \\ 2 & -1 & 2 & 3 \end{pmatrix}$$

To figure out what row interchanges are necessary we start Gauss elimination

$$\begin{pmatrix} 1 & 1 & -1 & 0 \\ 1 & 1 & 4 & 3 \\ 2 & -1 & 2 & 4 \\ 2 & -1 & 2 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 5 & 3 \\ 0 & -3 & 4 & 4 \\ 0 & -3 & 4 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & -3 & 4 & 4 \\ 0 & 0 & 5 & 3 \\ 0 & -3 & 4 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & -3 & 4 & 4 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Thus, we need $(E_2) \longleftrightarrow (E_3)$

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$PA = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 2 & -1 & 2 & 4 \\ 1 & 1 & 4 & 3 \\ 2 & -1 & 2 & 3 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & -3 & 4 & 4 \\ 0 & 0 & 5 & 3 \\ 0 & -3 & 4 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & -3 & 4 & 4 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Thus

$$PA = LU$$

where

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix} \quad U = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & -3 & 4 & 4 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Thus

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & -3 & 4 & 4 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

6.6 Special Types of Matrices

We consider 2 types of matrices for which Gauss elimination can be performed without row interchanges.

1) Strictly diagonally dominant matrices

<u>Def</u>: The $n \times n$ matrix A is called <u>strictly diagonally dominant</u> when

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$$

holds for each $i = 1, 2, \ldots, n$.

Ex: $A = \begin{pmatrix} 7 & 5 & -3 \\ 1 & 4 & 2 \\ -1 & -2 & 5 \end{pmatrix}$ $\quad B = \begin{pmatrix} -11 & 2 & 5 \\ -3 & -12 & 6 \\ 1 & 3 & 5 \end{pmatrix}$

A is <u>not</u> strictly diagonally dominant since

$$|7| \not> |5| + |3|$$

B is diagonally dominant

$$|-11| > |2| + |5|$$
$$|-12| > |-3| + |6|$$
$$|5| > |1| + |3|$$

Theorem 6.19. Let A - strictly diagonally dominant. Then

1) A is nonsingular

2) Gauss elimination can be performed on

$$A\vec{x} = \vec{b}$$

without row interchanges.

2) Positive definite matrices

<u>Def:</u> A matrix A is positive definite if A is symmetric and

$$\vec{x}^t A \vec{x} > 0$$

for every vector $\vec{x} \neq 0$.

Ex.

$$A = \begin{pmatrix} 4 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 4 \end{pmatrix}$$

$$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 4 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} =$$

$$= \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 4x_1 - 2x_2 \\ -2x_1 + 4x_2 - 2x_3 \\ -2x_2 + 4x_3 \end{pmatrix} =$$

$$= 4x_1^2 - 2x_1 x_2 - 2x_1 x_2 + 4x_2^2 - 2x_2 x_3 - 2x_2 x_3 + 4x_3^2$$

$$= 2x_1^2 + 2(x_1 - x_2)^2 + 2(x_2 - x_3)^2 + 2x_3^2 \geq 0$$

Theorem 6.21 If A is $n \times n$ positive definite, then

1) A is nonsingular
2) $a_{ii} > 0 \quad i=1,\dots,n$
3) The largest in absolute value element is along the diagonal

$$\max_{1 \le k,j \le n} |a_{kj}| \le \max_{1 \le i \le n} |a_{ii}|$$

Def: A <u>leading principal submatrix</u> of A is a matrix of the form

$$A_k = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{pmatrix}$$

for some $1 \le k \le n$.

Ex:
$$A = \begin{pmatrix} 4 & -2 & 0 \\ -2 & 4 & 0 \\ 0 & -2 & 4 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix}$$ is a leading submatrix

Theorem 6.23 Let A - symmetric
A - positive definite $\Leftrightarrow$ each of its principal submatrices has a positive determinant.

Ex: $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 4 & 2 \\ 0 & 2 & 2 \end{pmatrix}$

A-symmetric. Is A-positive definite.

$A_1 = 2$ $\det A_1 = 2 > 0$

$A_2 = \begin{pmatrix} 2 & -1 \\ -1 & 4 \end{pmatrix}$ $\det A_2 = 8 - 1 = 7 > 0$

$A_3 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 4 & 2 \\ 0 & 2 & 2 \end{pmatrix}$ $\det A_3 = 16 - 8 - 2 = 6 > 0.$

Thus A is positive definite.

Theorem 6.24 Let A-symmetric.
A-positive definite $\Leftrightarrow$ Gaussian elimination without row interchanges can be performed on $A\vec{x} = \vec{b}$ with all pivots positive.

Corollary. A-positive definite $\Leftrightarrow$ $A = LDL^t$ where L-lower triangular with 1's on the diagonal and D-diagonal with positive diagonal entries.

Procedure for $A = LDL^t$

1) Write $A = LU$
2) Write $U = DL^t$ where D has the diagonal entries of U.

Ex.
$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 4 & 2 \\ 0 & 2 & 2 \end{pmatrix} \to \begin{pmatrix} 2 & -1 & 0 \\ 0 & \frac{7}{2} & 2 \\ 0 & 2 & 2 \end{pmatrix}$$

$$\to \begin{pmatrix} 2 & -1 & 0 \\ 0 & \frac{7}{2} & 2 \\ 0 & 0 & \frac{6}{7} \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & \frac{4}{7} & 1 \end{pmatrix} \qquad U = \begin{pmatrix} 2 & -1 & 0 \\ 0 & \frac{7}{2} & 2 \\ 0 & 0 & \frac{6}{7} \end{pmatrix}$$

$$U = \begin{pmatrix} 2 & -1 & 0 \\ 0 & \frac{7}{2} & 2 \\ 0 & 0 & \frac{6}{7} \end{pmatrix} = \underbrace{\begin{pmatrix} 2 & 0 & 0 \\ 0 & \frac{7}{2} & 0 \\ 0 & 0 & \frac{6}{7} \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{4}{7} \\ 0 & 0 & 1 \end{pmatrix}}_{L^t}$$

Theorem 6.27 Let A-symmetric for which Gaussian elimination can be performed without row interchanges. Then
$$A = LDL^t$$
where L-lower triangular with 1's along the diagonal and D is diagonal.

Choleski's factorization:

Cor. 6.26 A - positive definite $\Leftrightarrow$ $A = LL^t$

L - lower triangular with nonzero diagonal entries.

3) Band matrices

<u>Def</u>: An $n \times n$ matrix is called a <u>band matrix</u> if integers p, q - exist $1 < p, q < n$ with the property that $a_{ij} = 0$ whenever

$$i + p \le j \quad \text{or} \quad j + q \le i$$

<u>Def</u>: The <u>bandwidth</u> of a band matrix is defined as

$$w = p + q - 1$$

Ex:
$$\begin{array}{c} \qquad\quad p \\ \begin{pmatrix} 1 & 5 & 6 & 0 \\ -1 & 2 & 7 & 8 \\ 0 & -2 & 3 & 9 \\ 0 & 0 & -3 & 4 \end{pmatrix} \\ \qquad\quad q \end{array}$$

$p = 3 \quad q = 2$

$w = 3 + 2 - 1 = 4$

Typically $p = q$

Ex $\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$ $p=q=2$

$w=2+2-1=3$

At most 3 elements on a row are nonzero.

Def: Matrices with $p=q=2$ and bandwidth $w=3$ are called <u>tridiagonal</u>

They have the form

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 & \cdots & & 0 \\ a_{21} & a_{22} & a_{23} & \ddots & & \vdots \\ 0 & a_{32} & a_{33} & a_{34} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & a_{n-1,n} \\ 0 & \cdots & \cdots & 0 & a_{n,n-1} & a_{nn} \end{pmatrix}$$

Crout's factorization for tridiagonal matrices

$$A = LU$$

where L - lower-triangular, U - upper-triang.

$$L = \begin{pmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \ddots & \vdots \\ 0 & \ddots & \ddots & 0 \\ 0 & \cdots 0 & l_{n,n-1} & l_{nn} \end{pmatrix} \quad U = \begin{pmatrix} 1 & u_{12} & 0 & \cdots & 0 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & u_{n-1,n} \\ 0 & \cdots & & 0 & 1 \end{pmatrix}$$

The nonzero entries in L, U are determined from

$$a_{11} = l_{11}$$
$$a_{i,i-1} = l_{i,i-1} \quad i = 2, \ldots, n$$
$$a_{ii} = l_{i,i-1} u_{i-1,i} + l_{ii} \quad i = 2, \ldots, n$$

$$a_{i,i+1} = l_{ii} u_{i,i+1} \quad i = 1, 2, \ldots, n-1$$

Ex: Find Crout's LU factorization of

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 2 & 4 & 1 \\ 0 & 2 & 5 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 2 & \frac{10}{3} & 0 \\ 0 & 2 & \frac{22}{5} \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{3} & 0 \\ 0 & 1 & \frac{3}{10} \\ 0 & 0 & 1 \end{pmatrix}$$

$l_{11} = 3 \qquad l_{21} = a_{21} = 2 \qquad l_{32} = a_{32} = 2$

$$a_{12} = l_{11} u_{12} \Rightarrow u_{12} = \frac{a_{12}}{l_{11}} = \frac{1}{3}$$

$$a_{22} = l_{21} u_{12} + l_{22} \Rightarrow 4 = 2 \cdot \frac{1}{3} + l_{22} \Rightarrow l_{22} = \frac{10}{3}$$

$$l_{32} = a_{32} = 2$$

$$a_{23} = l_{22} u_{23} \rightarrow u_{23} = \frac{a_{23}}{l_{22}} = \frac{1}{\frac{10}{3}} = \frac{3}{10}$$

$$a_{33} = l_{32} u_{23} + l_{33} \rightarrow 5 = 2 \cdot \frac{3}{10} + l_{33} \rightarrow l_{33} = \frac{22}{5}$$