6.2 Pivoting Strategies

1) Problems with small pivots

We saw that if $a_{kk} = 0$ at some point we have to find an element in the same column $a_{pk} \neq 0$ and we have to interchange

$$(E_k) \leftrightarrow (E_p)$$

If $a_{kk} \neq 0$ but small (in abs. value) then the multipliers

$$m_{jk} = \frac{a_{jk}}{a_{kk}}$$

will be much larger than one. Round off errors introduced in the computation of some of the elements a_{ke} is multiplied by m_{jk} which compounds the original error.

Also, in the backward substitution

$$x_k = \frac{b_k - a_{kn}x_n - \dots - a_{kk+1}x_{k+1}}{a_{kk}}$$

with small value of a_{kk} any error in the numerator can be dramatically increased.

Ex. Use Gaussian elimination and a three-digit chopping arithmetic to solve the following linear system

$$0.03\,x_1 + 58.9\,x_2 = 59.2$$
$$5.31\,x_1 - 6.10\,x_2 = 47.0$$

Actual solution $(10, 1)$.

Multiply the first row by

$$m_{21} = \frac{5.31}{0.03} = 177$$

Since $a_{11} = 0.03$ - small, m_{21} - large.
Multiplying the first row by 177

Actual: $5.31\,x_1 + 10425.3\,x_2 = 10478.4$

Chopped $5.31\,x_1 + 10400\,x_2 = 10400$

Multiplying by (-1) and adding to second row

Actual $-10431.4\,x_2 = -10431.4 \Rightarrow x_2 = 1$

Chopped $-10400\,x_2 = -10300$

$\Rightarrow x_2 = 0.990$

Performing back substitution on the chopped system

$$0.03x_1 + 58.9x_2 = 59.2$$
$$10400\,x_2 = 10300$$

$$x_1 = \frac{59.2 - 58.9 * 0.99}{0.03} = \frac{59.2 - 58.3}{0.03} =$$
$$= \frac{0.9}{0.03} = 30 \quad (\text{Exact value } x_1 = 10)$$

Thus the small error of -0.01 in x_2 is multiplied by
$$\frac{58.9}{0.03} = 1963$$
which ruins the approximation of x_1.

This example shows what difficulties arise if the pivot element a_{kk} is small, relative to the other entries.

2) Partial Pivoting

To avoid the problem described above, pivoting is performed by selecting a larger element a_{pq} for pivot and interchanging rows $(E_k) \leftrightarrow (E_p)$ followed by interchanging

k^{th} and q^{th} columns if necessary.

Strategy: Select an element in the column below the diagonal and has largest absolute value, that is determine smallest $p > k$ such that

$$|a_{pk}| = \max_{k \le i \le n} |a_{ik}|$$

and perform $(E_k) \leftrightarrow (E_p)$. In this case no column interchange.

Ex: Reconsider the system

$$0.03x_1 + 58.9x_2 = 59.2$$
$$5.31x_1 - 6.10x_2 = 47.0$$

Instead of using 0.03 as a pivot we look in the column below it for the largest in abs. value coefficient. That is 5.31. We will use that as pivot. Thus, we begin by interchanging $(E_1) \leftrightarrow (E_2)$

$$5.31x_1 - 6.10x_2 = 47$$
$$0.03x_1 + 58.9x_2 = 59.2$$

Multiply the first row by

$$m_{21} = \frac{0.03}{5.31} = 0.00564$$

Multiplying the first row by m_{21} we get

Chopped: $0.03\,x_1 - 0.0344\,x_2 = 0.265$

Now multiplying this by (-1) and adding to second equation produces the system

$$5.31\,x_1 - 6.10\,x_2 = 47$$
$$58.9\,x_2 = 58.9$$

From here $x_2 = 1$. Backward substitution leads to

$$x_1 = \frac{47 + 6.10}{5.31} = \frac{53.1}{5.31} = 10$$

That is, we get the exact solution.

Def: The partial pivoting or maximal column pivoting is a procedure which places as pivot the largest in magnitude element in the pivot column.

Note: In this case $|m_{ij}| \le 1$.

Although this strategy is sufficient for mos linear systems, situations arise that this is inadequate.

3) Scaled partial pivoting

Ex: Consider the system

$$30.0\, x_1 + 58900\, x_2 = 59200$$
$$5.31\, x_1 - 6.10\, x_2 = 47$$

This is the same system as before except the first equation has been multiplied by 10^3.

Performing partial pivoting we find that 30 is the maximal element.

Thus, we multiply the first equation by

$$m_{21} = \frac{5.31}{30.0} = 0.177$$

Chopping: $5.31\, x_1 + 10400.0\, x_2 = 10400.0$

Subtracting from the second eqn.

$$30.0\, x_1 + 58900\, x_2 = 59200$$
$$-10400\, x_2 = -10300$$
$$\Rightarrow x_2 = 0.99$$

$$x_1 = \frac{59200 - 58900\cdot(0.99)}{30} = 30$$

The same result as in the first example
The exact solution of this system is $(10, 1)$.

Thus, the partial pivoting does not help with this example.

<u>Def</u>: <u>Scaled partial</u> pivoting or <u>scaled-column pivoting</u> is a procedure which chooses as a pivot the element that is largest relative to the entries in its row.

<u>Procedure:</u>

1) Define scale factors s_i for each row a

$$s_i = \max_{1\le j\le n} |a_{ij}|$$

(s_i is the magnitude of the largest eleme[nt] of the row).

Note: If $s_i = 0$ then the system has a row of zeroes and there is no unique solution.

2) Choose the least p with

$$\frac{|a_{pk}|}{s_p} = \max_{1 \le j \le n} \frac{|a_{jk}|}{s_j}$$

That is choose the row p such that the element in the column below the pivot has maximal value after being divided by the row factor s_j, and perform interchange

$$(E_k) \leftrightarrow (E_p)$$

Note: The factors $s_1, \ldots, s_n$ are computed only once at the beginning and must be interchanged when rows are interchanged.

Ex: Apply scaled pivoting to

$$30.0\,x_1 + 58900\,x_2 = 59200$$
$$5.31\,x_1 - 6.10\,x_2 = 47$$

The scale factors are

$$s_1 = 58900$$
$$s_2 = 6.1$$

Consequently, performing scaled pivoting we compare the scaled elements in the pivot column:

$$\frac{30}{58900} = 5.09 * 10^{-4} \qquad \frac{5.31}{6.10} = 0.87049$$

Thus, the pivot should be 5.31.
Interchanging

$$(E_1) \leftrightarrow (E_2)$$

we get the system

$$\begin{aligned} 5.31\, x_1 - 6.10\, x_2 &= 47 \\ 30.0\, x_1 + 58900\, x_2 &= 59200 \end{aligned}$$

Using Gauss-elimination and 3-digit chopping arithmetic we get the exact solution:

$$m_{21} = \frac{30.0}{5.31} = 5.65$$

Multiplying the first row by m_{21}

Chopping $\quad 30.0\, x_1 - 34.4\, x_2 = 265$
and subtracting from the second equation we get the system

$$\begin{aligned} 5.31\, x_1 - 6.10\, x_2 &= 47 \\ 58900\, x_2 &= 58900 \end{aligned}$$

Thus, $x_2 = 1$, $x_1 = 10$.