

5.6 Multistep Methods

1) Definition

The methods discussed so far need

$w_i \rightarrow w_{i+1}$
 w_i to compute w_{i+1} . Such methods are called one-step methods. Even if these methods use the value of the solution in an intermediate point this information is not retained.

Since the approximate solution at t_0, \dots, t_i is available it is reasonable to develop methods which use this information.

Methods, using the approximation at more than one previous mesh points are called multistep methods.

Def An m-step multistep method for solving the IVP

$$\begin{aligned} y' &= f(t, y) & a \leq t \leq b \\ y(a) &= \alpha \end{aligned}$$

has a difference equation for finding the approximation w_{i+1} at the point t_{i+1} represented by the following equation

$$w_{i+1} = a_1 w_i + a_2 w_{i-1} + \dots + a_m w_{i-m+1}$$

$$(*) \quad + h [b_0 f(t_{i+1}, w_{i+1}) + b_1 f(t_i, w_i) + \dots +$$

$$b_m f(t_{i-m+1}, w_{i-m+1})]$$

where $i = m-1, m, \dots, N$, $h = \frac{b-a}{N}$,

$a_1, \dots, a_m, b_0, \dots, b_m$ are constants and the starting values

$w_0 = d_0, w_1 = d_1, \dots, w_{m-1} = d_{m-1}$ are specified.

Def: When $b_0 = 0$ the method is called explicit or open.

Note: Equation (*) gives w_{i+1} explicitly in terms of previously computed values

Def: When $b_0 \neq 0$ the method is called implicit or closed.

Note: w_{i+1} occurs at both sides of (*).

2) Examples of Multistep methods

a) Adam-Bashforth Two-Step Explicit Method
 starting values $\rightarrow w_0 = \alpha \quad w_1 = d_1$

$$w_{i+1} = w_i + \frac{h}{2} [3f(t_i, w_i) - f(t_{i-1}, w_{i-1})]$$

$$i = 1, 2, \dots, N-1$$

The local truncation error is

$$\tau_{i+1}(h) = \frac{5}{12} y'''(\xi_i) h^2 \quad \xi_i \in (t_{i-1}, t_i)$$

b) Adam-Bashforth Three-step Explicit Method

$$w_0 = \alpha \quad w_1 = d_1 \quad w_2 = d_2$$

$$w_{i+1} = w_i + \frac{h}{12} [23f(t_i, w_i) - 16f(t_{i-1}, w_{i-1}) + 5f(t_{i-2}, w_{i-2})]$$

$$i = 2, 3, \dots, N-1$$

The local truncation error is

$$\tau_{i+1}(h) = \frac{3}{8} y^{(4)}(\xi_i) h^3 \quad \xi_i \in (t_{i-2}, t_{i+1})$$

c) Adam-Bashforth Four-step Explicit Method

$$w_0 = \alpha \quad w_1 = d_1 \quad w_2 = d_2 \quad w_3 = d_3$$

$$w_{i+1} = w_i + \frac{h}{24} [55f(t_i, w_i) - 59f(t_{i-1}, w_{i-1}) + 37f(t_{i-2}, w_{i-2}) - 9f(t_{i-3}, w_{i-3})]$$

where $i = 3, 4, \dots, N-1$. The local truncation error

$$\tau_{i+1}(h) = \frac{251}{720} y^{(5)}(\xi_i) h^4$$

d) Adams-Moulton Two-step Implicit Method

$$w_0 = \alpha \quad w_1 = \alpha_1$$
$$w_{i+1} = w_i + \frac{h}{2} [5f(t_{i+1}, w_{i+1}) + 8f(t_i, w_i) - f(t_{i-1}, w_{i-1})]$$

where $i = 1, 2, \dots, N-1$. The local truncation error

$$\tilde{\tau}_{i+1}(h) = -\frac{1}{24} y^{(4)}(\xi_i) h^3 \quad \xi_i \in (t_{i-1}, t_{i+1})$$

e) Adams-Moulton Three-step Implicit Method

$$w_0 = \alpha \quad w_1 = \alpha_1 \quad w_2 = \alpha_2$$

$$w_{i+1} = w_i + \frac{h}{24} [9f(t_{i+1}, w_{i+1}) + 19f(t_i, w_i) - 5f(t_{i-1}, w_{i-1}) + f(t_{i-2}, w_{i-2})]$$

$i = 2, 3, \dots, N-1$. The local truncation error

$$\tilde{\tau}_{i+1}(h) = -\frac{19}{720} y^{(5)}(\xi_i) h^4 \quad \xi_i \in (t_{i-2}, t_{i+1})$$

Remarks:

- 1) The starting values must be specified. Generally one takes $w_0 = \alpha$ and generates the remaining values by a one-step method (usually Runge-Kutta methods)
- 2) To apply an implicit method one typically has to solve a nonlinear equation.

Ex: Applying the 3-step Adam-Moulton method to

$$\begin{aligned} y' &= e^y & 0 \leq t \leq 0.2 \\ y(0) &= 1 \end{aligned}$$

$$w_{i+1} = w_i + \frac{h}{24} [9e^{w_{i+1}} + 19e^{w_i} - 5e^{w_{i-1}} + e^{w_{i-2}}]$$

Given w_{i-2}, w_{i-1}, w_i , to find w_{i+1} , we have to solve nonlinear equation

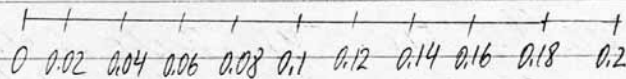
$$w_{i+1} = g(w_{i+1})$$

where

$$g(w_{i+1}) = w_i + \frac{h}{24} [9e^{w_{i+1}} + 19e^{w_i} - 5e^{w_{i-1}} + e^{w_{i-2}}]$$

One way to deal with that is to apply a functional iteration. We may use w_i as an initial approximation of w_{i+1} .

For this problem, say $h = 0.02$.



The exact solution is $y(t) = 1 - \ln(1 - et)$

Take

$$w_0 = 1 \quad w_1 = 1.0558993 \quad w_2 = 1.1157$$

Thus, w_3 satisfies

$$w_3 = 1.1151 + 0.0075e^{w_3} + 0.0385775636$$

$$w_3 = 1.153677564 + 0.0075e^{w_3}$$

We may apply iteration with $w_3^0 = 1.1151$ to solve this equation, or Newton's method. That will compute w_3 . Next, we will obtain similar equation for w_4 and so on.

- 3.) Notice that the local truncation error of a 2-step explicit method is $O(h^2)$ while of an Implicit method is $O(h^3)$. Thus, the corresponding Implicit method is an order of magnitude more accurate. At the same time it takes more steps to compute since at each step we have to solve a nonlinear eqn.

Actually, one typically compares an m -step Adams-Bashforth explicit method to $(m-1)$ -step Adams-Moulton implicit methods. Both involve m evaluations of f per step and both have local truncation error $O(h^m)$.

In general implicit methods have greater stability and smaller roundoff errors. See table 5.11.

3) Predictor-Corrector Methods

In practice since implicit methods are more accurate but hard to compute with, they are not used alone.

Implicit methods are used to improve approximations obtained by an explicit method. The combination of an explicit and implicit method is called a predictor-corrector method.

The explicit method is used to predict the approximation and the implicit method corrects that prediction

Ex: To solve the IVP

$$y' = e^y \quad 0 \leq t \leq 0.2$$

$$y(0) = 1$$

we may use a 4-step explicit Adams-Bashforth method as a predictor

Thus

$$w_0 = 1 \quad w_1 = 1.0558993 \quad w_2 = 1.1157$$

$$w_3 = 1.178047$$

we compute $w_4^{(0)}$ using explicit 4-step Adam-Bashforth method as predictor

$$w_4^{(0)} = w_3 + \frac{h}{24} [55e^{w_3} - 59e^{w_2} + 37e^{w_1} - 9e]$$

This approximation is improved by inserting $w_4^{(0)}$ in the right-hand side of a 3-step implicit Adams-Moulton method using this method as a corrector.

$$w_4^{(1)} = w_3 + \frac{h}{24} [9e^{w_4^{(0)}} + 19e^{w_3} - 5e^{w_2} + e^{w_1}]$$

Finally, we set $w_4 = w_4^{(1)}$ and we repeat the procedure to compute w_5, \dots, w_N .