6.3 Linear Algebra and Matrix Inversion

Matrices are a convinient method for expressing and manipulating linear systems. In this section we consider some algebra involving matrices.

1) Sum and product of matrices.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

A is $m \times n$ matrix.

Def: Two matrices A, B are equal if they have the same number of rows and columns and

$$a_{ij} = b_{ij} \qquad \begin{matrix} i = 1, \ldots, m \\ j = 1, \ldots, n \end{matrix}$$

Ex: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

since they differ in dimension.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Ex. Find the value(s) of x so that the matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} x^2 & 0 \\ 0 & x^2 \end{pmatrix}$$

are equal.

Need $x^2 = 1 \Rightarrow x = \pm 1$.

Def: If A, B are $m \times n$ matrices, then the <u>sum</u> $A+B$ is an $m \times n$ matrix whose entries are $a_{ij} + b_{ij}$ $\quad i = 1, \dots, m$, $j = 1, \dots, n$

Ex. $$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \quad B = \begin{pmatrix} -2 & 3 & -5 \\ -3 & 7 & -11 \end{pmatrix}$$

$$A + B = \begin{pmatrix} -1 & 5 & -2 \\ 1 & 12 & -5 \end{pmatrix}$$

Def: If A is an $m \times n$ matrix and λ is a real number, then the <u>scalar multiplication</u> of λ and A, denoted λA, is an $m \times n$ matrix whose entries are λa_{ij} $\quad i = 1, \dots, m$, $j = 1, \dots, n$

Ex: $A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 3 & -3 \end{pmatrix}$ $\lambda = 5$

$$\lambda A = 5A = \begin{pmatrix} 5\cdot 1 & 5\cdot(-1) & 5\cdot(2) \\ 5\cdot(-2) & 5\cdot 3 & 5(-3) \end{pmatrix} = \begin{pmatrix} 5 & -5 & 10 \\ -10 & 15 & -15 \end{pmatrix}$$

2) $B = \begin{pmatrix} 2 & -4 & -8 \\ 10 & -12 & 16 \end{pmatrix}$ $\lambda = \frac{1}{2}$

$$\lambda B = \frac{1}{2} B = \begin{pmatrix} \frac{1}{2}\cdot 2 & \frac{1}{2}(-4) & \frac{1}{2}(-8) \\ \frac{1}{2}\cdot 10 & \frac{1}{2}(-12) & \frac{1}{2}(16) \end{pmatrix} = \begin{pmatrix} 1 & -2 & -4 \\ 5 & -6 & 8 \end{pmatrix}$$

Def: $-A = (-1)\cdot A$

Def: $0 = \begin{pmatrix} 0 & 0 & 0 \cdots & 0 \\ \vdots & & & \\ 0 & 0 & 0 \cdots & 0 \end{pmatrix}$, $m \times n$ matrix whose entries are all 0.

The matrix addition and scalar multiplication satisfies usual properti (as real numbers).

Ex: $A+B = B+A$, $(A+B)+C = A+(B+C)$

$\lambda(A+B) = \lambda A + \lambda B$ $(\lambda+\mu)A = \lambda A + \mu A$

(see the book).

Def: Let A be $n \times m$ matrix and B be $m \times p$ matrix. The matrix product of A and B, denoted AB, is a $n \times p$ matrix C whose entries c_{ij} are

$$c_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{im} b_{mj}$$
$$i = 1, \ldots, n$$
$$j = 1, \ldots, p.$$

$$AB = \begin{pmatrix} \\ a_{i1} \cdot a_{i2} \cdots a_{im} \\ \\ \end{pmatrix} \begin{pmatrix} & b_{1j} & \\ & b_{2j} & \\ & \vdots & \\ & b_{mj} & \end{pmatrix} = \begin{pmatrix} & & \\ & c_{ij} & \\ & & \end{pmatrix}$$

Note: This is why the number of rows of A must equal the number of colums of B for AB to be defined.

Ex:
$$\underset{3 \times 3}{\begin{pmatrix} -1 & 2 & -2 \\ 1 & 3 & 5 \\ 0 & -3 & -1 \end{pmatrix}} \underset{3 \times 2}{\begin{pmatrix} 0 & 3 \\ -1 & 0 \\ 2 & -2 \end{pmatrix}} = \underset{3 \times 2}{\begin{pmatrix} -6 & 1 \\ 7 & -7 \\ 1 & 2 \end{pmatrix}}$$

Note: BA cannot be computed.

Thus, $AB \neq BA$.

Def: A square matrix has the same number of rows as columns ($n \times n$ matrix).

Def: A *diagonal* matrix is a square matrix $D = (d_{ij})$ with $d_{ij} = 0$ whenever $i \neq j$

D - diagonal, if the off diagonal elements are zero.

Ex: $\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$ Note: diagonal elements may be zero.

Def: The *identity matrix of order n* I_n is a diagonal matrix whose entries along the diagonal are ones.

$$I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Def: An upper-triangular $n \times n$ matrix $U = (u_{ij})$ has the entries

$$u_{ij} = 0 \quad \text{for } i = j+1, \ldots, n \qquad j = 1, \ldots, n.$$

Ex: $$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$$

Def: A <u>lower-triangular</u> matrix $L=(l_{ij})$ is an $n\times n$ matrix whose entries

$$l_{ij}=0 \quad \begin{array}{l} i=1,\cdots,j-1 \\ j=1,\cdots,n \end{array}$$

Ex. $L=\begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{pmatrix}$

<u>Remark:</u> $AI_n = I_n A$ for any A $n\times n$.

<u>Theorem 6.9.</u>
A $n\times m$ matrix
B $n\times k$ matrix
C $k\times p$ matrix
D $m\times k$ matrix

(a) $A(BC)=(AB)C$
(b) $A(B+D)=AB+AD$
(c) $I_m B = B \quad BI_k = B$
(d) $\lambda(AB)=(\lambda A)B = A(\lambda B)$

Note: $AB \neq BA$.

2) The Inverse of a matrix and linear systems.

Def: An $n \times n$ matrix A is said to be nonsingular (or invertible) if an $n \times n$ matrix B exists such that

$$AB = BA = I$$

The matrix B is called the inverse of and is denoted by A^{-1}.

Def: A matrix without an inverse is callea singular.

Theorem 6.11. For any nonsingular matr. A

(a) A^{-1} is unique

(b) A^{-1} is also nonsingular and $(A^{-1})^{-1} = A$

(c) If B is also nonsingular $n \times n$ matrix then

$$(AB)^{-1} = B^{-1}A^{-1}$$

Ex:

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & -1 \\ 3 & 1 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{5}{8} & -\frac{1}{8} & -\frac{1}{8} \\ \frac{1}{8} & -\frac{5}{8} & \frac{3}{8} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & -1 \\ 3 & 1 & 1 \end{pmatrix}\begin{pmatrix} -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{5}{8} & -\frac{1}{8} & -\frac{1}{8} \\ \frac{1}{8} & -\frac{5}{8} & \frac{3}{8} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$-\frac{1}{4} + \frac{10}{8} = -\frac{2}{8} + \frac{10}{8} = 1 \quad \frac{1}{4} - \frac{2}{8} \quad \frac{1}{4} - \frac{2}{8}$

$-\frac{2}{4} + \frac{5}{8} - \frac{1}{8} = -\frac{1}{2} + \frac{1}{2} \quad \frac{2}{4} - \frac{1}{8} + \frac{5}{8} \quad \frac{2}{4} - \frac{1}{8} - \frac{3}{8}$

It can be shown similarly that $BA = I$. Thus, $B = A^{-1}$.

- Representation of linear systems in matrix form.

Consider the system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

can be rewritten as

$$A\vec{x} = \vec{b}$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \quad \vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

Ex: Consider the linear system

$$\begin{aligned} x_1 + 2x_2 \quad &= 2 \\ 2x_1 + x_2 - x_3 &= -4 \\ 3x_1 + x_2 + x_3 &= -2 \end{aligned}$$

We can rewrite this as

$$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & -1 \\ 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ -2 \end{pmatrix}$$

or

$$A\vec{x} = \vec{b}$$

Multiplying this equations with A^{-1}

$$A^{-1}A\vec{x} = A^{-1}\vec{b}$$

$$I\vec{x} = A^{-1}b$$

$$\vec{x} = A^{-1}b$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{5}{8} & -\frac{1}{8} & -\frac{1}{8} \\ \frac{1}{8} & -\frac{5}{8} & \frac{3}{8} \end{pmatrix} \begin{pmatrix} 2 \\ -4 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 2 \end{pmatrix}$$

solution

Thus, if we know A^{-1}, solving the system is easy.

To compute the inverse of a matrix we write an extended augmented matrix

$$\left(A \mid I \right)$$

Then we perform operations (I, II, III) on A to get I and the same operations on I. Finally we get

$$\left(I \mid A^{-1} \right)$$

Ex:
$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 2 & 1 & -1 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \to$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -3 & -1 & -2 & 1 & 0 \\ 0 & -5 & 1 & -3 & 0 & 1 \end{array}\right) \to$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 15 & 5 & 10 & -5 & 0 \\ 0 & -15 & 3 & -9 & 0 & 3 \end{array}\right) \to$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 15 & 5 & 10 & -5 & 0 \\ 0 & 0 & 8 & 1 & -5 & 3 \end{array}\right) \to$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & \frac{1}{8} & -\frac{5}{8} & \frac{3}{8} \end{array}\right) \to$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & \frac{15}{8} & -\frac{3}{8} & -\frac{3}{8} \\ 0 & 0 & 1 & \frac{1}{8} & -\frac{5}{8} & \frac{3}{8} \end{array}\right) \longrightarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{5}{8} & -\frac{1}{8} & -\frac{1}{8} \\ 0 & 0 & 1 & \frac{1}{8} & -\frac{5}{8} & \frac{3}{8} \end{array}\right) \longrightarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{2}{8} & \frac{2}{8} & \frac{2}{8} \\ 0 & 1 & 0 & \frac{5}{8} & -\frac{1}{8} & -\frac{1}{8} \\ 0 & 0 & 1 & \frac{1}{8} & -\frac{5}{8} & \frac{3}{8} \end{array}\right)$$

$$\Rightarrow A^{-1} = \begin{pmatrix} -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{5}{8} & -\frac{1}{8} & -\frac{1}{8} \\ \frac{1}{8} & -\frac{5}{8} & \frac{3}{8} \end{pmatrix}$$

Computationally, finding the inverse is equivalent to solving the matrix equation

$$AB = I$$

where (b_{ij}) are unknown or, equivalently, n systems.

Def: The transpose of an $n \times m$ matrix $A = (a_{ij})$ is the matrix A^t where the i^{th} column of A^t is the same as the i^{th} row of A.

Ex $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ $A^t = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$

<u>Def:</u> A square matrix A is called symmetric if

$$A^t = A$$

Ex: $A = \begin{pmatrix} 1 & -2 & 5 \\ -2 & 2 & \frac{1}{3} \\ 5 & \frac{1}{3} & 3 \end{pmatrix}$ - symmetric matrix

$$A^t = \begin{pmatrix} 1 & -2 & 5 \\ -2 & 2 & \frac{1}{3} \\ 5 & \frac{1}{3} & 3 \end{pmatrix} = A$$

Theorem 6.13. The following are valid

(a) $(A^t)^t = A$

(b) $(A+B)^t = A^t + B^t$

(c) $(AB)^t = B^t A^t$

(d) If A^{-1} - exists, then

$$(A^{-1})^t = (A^t)^{-1}$$