4.4 Composite Numerical Integration

If the interval $[a, b]$ is too large, using Simpson's, trapezoidal or midpoint rule can give a significant error

Ex: Use Simpson's rule to evaluate

$$\int_{-2}^{2} x^3 e^x \, dx \qquad h = \frac{b-a}{2} = 2$$

$$\int_{-2}^{2} x^3 e^x \, dx \approx \frac{2}{3}\left[f(-2) + 4f(0) + f(2)\right]$$

$$= \frac{2}{3}\left[(-2)^3 e^{-2} + 2^3 e^2\right] =$$

$$= \frac{2}{3}\left[-1.082682266 + 59.11244879\right]$$

$$= \frac{2}{3}\, 58.02976653 = 38.68651102$$

The exact value is 19.92085296

Thus the error is big

This error can be reduced. Suppose we write

$$\int_{-2}^{2} x^3 e^x \, dx = \int_{-2}^{0} x^3 e^x \, dx + \int_{0}^{2} x^3 e^x \, dx$$

and apply Simpson's rule to approximate each of the integrals

$$\int_{-2}^{0} x^3 e^x \, dx = \frac{1}{3}\left[(-2)^3 e^{-2} - 4e^{-1} + 0\right]$$

$$= \frac{1}{3}\left[-1.082682266 - 1.471517765\right.$$

$$= \frac{1}{3}\left[-2.554200031\right]$$

$$= -0.8514000102$$

$$\int_{0}^{2} x^3 e^x \, dx = \frac{1}{3}\left[0 + 4e^1 + (2)^3 e^2\right] =$$

$$= \frac{1}{3}\left[69.98557611\right]$$

$$= 23.32852537$$

$$\Rightarrow \int_{-2}^{2} x^3 e^x \, dx \approx 22.47712536$$

Next, we write

$$\int_{-2}^{-1} x^3 e^x dx + \int_{-1}^{0} x^3 e^x dx + \int_{0}^{1} x^3 e^x dx + \int_{1}^{2} x^3 e^x dx$$

$$\int_{-2}^{-1} x^3 e^x \, dx = \frac{1}{6}\left[(-2)^3 e^{-2} + 4(-1.5)^3 e^{-1.5} + (-1)^3 e^{-1}\right]$$

$$= \frac{1}{6}\left[-4.462818869\right] = -0.7438031448$$

$$\int_{-1}^{0} x^3 e^x \, dx = \frac{1}{6}\left[(-1)^3 e^{-1} + 4(-0.5)^3 e^{-0.5} + 0\right] =$$

$$= \frac{1}{6}\left[-0.671144771\right] = -0.1118574618$$

$$\int_{0}^{1} x^3 e^x \, dx = \frac{1}{6}\left[0 + 4(0.5)^3 e^{0.5} + 1^3 e^1\right] =$$

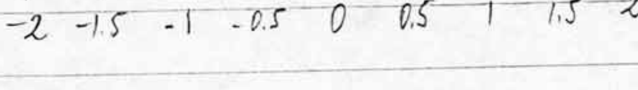
$$= \frac{1}{6}\left[3.542642464\right] = 0.5904404106$$

$$\int_{1}^{2} x^3 e^x \, dx = \frac{1}{6}\left[1 \cdot e^1 + 4(1.5)^3 e^{1.5} + 2^3 e^2\right] =$$

$$= \frac{1}{6}\left[122.3335331\right] = 20.38892218$$

$$\Rightarrow \int_{-2}^{2} x^3 e^x \, dx \approx 20.12370198$$

Error = 0.2028490222



Adding the formulas we have

$$\int_{-2}^{2} x^3 e^x dx = \frac{1}{6}[f(2)+4f(-1.5)+f(-1)+f(-1)+4f(-0.5)$$

$$+f(0)+f(0)+4f(0.5)+f(1)+f(1)+4f(1.5)$$

$$+f(2)]$$

$$=\frac{1}{6}[f(2)+4(-1.5)+2f(-1)+4f(-0.5)+2f(0)+4f(0.5)$$

$$+2f(1)+4f(1.5)+f(2)]$$

Generalizing this procedure

1) Choose n-even
2) Subdivide the interval $[a,b]$ into n subintervals
3) Applying Simpson's rule on each consequitive pair of subintervals. Thus we obtain

The Composite Simpson's rule: Let $h=\frac{b-a}{n}$

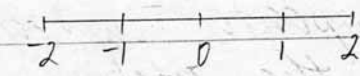
$$\int_a^b f(x)dx = \frac{h}{3}[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+\dots$$

$$+4f(x_{n-1})+f(x_n)] - \frac{b-a}{180} h^4 f^{(4)}(\xi) \quad a<\xi<b$$

Ex. Use the Composite Simpson's rule to
\# 2 6/203 approximate the integral

$$\int_{-2}^{2} x^3 e^x \, dx \quad \text{with } n=4$$

Solution: $n=4 \Rightarrow h = \frac{2-(-2)}{4} = \frac{4}{4} = 1$



$$\int_{-2}^{2} x^3 e^x \, dx = \frac{1}{3}\left[f(-2)+4f(-1)+2f(0)+4f(1)+f(2)\right] =$$

$$= \frac{1}{3}\left[(-2)^3 e^{-2} + 4(-1)^3 e^{-1} + 2\cdot 0 + 4(1)^3 e^1 + (2)^3 e^2\right] =$$

$$= \frac{1}{3}\left[67.431376\right] = 22.47712536$$

We can use the same procedure to derive Composite Trapezoidal, Composite Simpson 3/8 rule, Composite Midpoint rule

Composite Trapezoidal Rule

1) Divide the interval into n subintervals
$h = \frac{b-a}{n}$
2) Apply Trapezoidal Rule in each subinterval
Thus, we obtain

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(x_n)\right]$$
$$-\frac{b-a}{12}h^2 f''(\xi)$$
where $a < \xi < b$

Ex. #1(b)/203 Use the Composite Trapezoidal rule to approximate the integral
$$\int_{-2}^{2} x^3 e^x\,dx \quad \text{with } n=4$$

Solution: $h = \frac{2-(-2)}{4} = 1$
$$\int_{-2}^{2} x^3 e^x\,dx = \frac{1}{2}\left[(-2)^3 e^{-2} + 2(-1)^3 e^{-1} + 2\cdot 0 + 2(1)^3 e^1 + (2)^3 e^2\right] =$$
$$= \frac{1}{2}\left[62.7305713\right] = 31.36528565$$

Error $= 11.44443269$ (bad)

Note: The Composite Trapezoidal rule gives much worse approximation than the Composite Simpson's rule with the same number of subintervals. With Composite Trapezoidal we need to use much more subintervals to achive reasonable approximation

The Composite Simpson's 3/8 rule.

1) Choose n - divisible by 3
2) Subdivide the interval $[a,b]$ into n subintervals

$$h = \frac{b-a}{n}$$

3) Apply Simpson's 3/8 rule on each consequitive set of 3 subintervals. Thus, we obtain

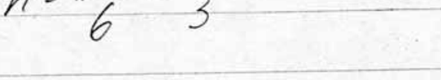
$$\int_a^b f(x)\,dx = \frac{3h}{8}\big(f(x_0)+3f(x_1)+3f(x_2)+2f(x_3)+\dots+ 3f(x_{n-1})+f(x_n)\big] - \frac{b-a}{80}h^4 f^{(IV)}(\xi)$$

where $a < \xi < b$

Ex. Use Composite Simpson's 3/8 rule to approximate

$$\int_0^2 \frac{2}{x^2+4}\,dx \quad \text{with } n=6$$

Solution: $h = \frac{2}{6} = \frac{1}{3}$



$$\int_0^2 \frac{2}{x^2+4}\,dx = \frac{3}{8}\cdot\frac{1}{3}\Big[\frac{1}{2} + 3\cdot\frac{2}{(\frac{1}{3})^2+4} + 3\frac{2}{(\frac{2}{3})^2+4} + 2\frac{2}{1+4}$$

$$+ 3\cdot\frac{2}{(1\frac{1}{3})^2+4} + 3\frac{2}{(1\frac{2}{3})^2+4} + \frac{2}{2^2+4}\Big] =$$

$$= \frac{1}{8}\big[6.2831669\big] = 0.7853958624$$

Exact value $= \frac{\pi}{4} = 0.7853981634$

Error $= 2.3 \cdot 10^{-6}$

The Composite Simpson's 3/8 rule with $n=6$ is fairly accurate.

The Composite Midpoint rule.

1) Choose n - even
2) Subdivide the interval (a,b) into $n+2$ subintervals. The first point being $a = x_{-1}$ and the last $b = x_{n+1}$.
3) Apply Midpoint rule on each pair of intervals. Thus we obtain the formula

$$\int_a^b f(x)\,dx = 2h\,[f(x_0) + f(x_2) + \dots + f(x_n)] + \frac{b-a}{6}h^2 f''(\xi)$$

where $a < \xi < b$

Theorem 4.6. Let

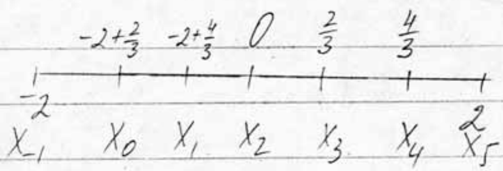
1) f have 2 continuous derivatives on $[a,b]$
2) n be even
3) $h = \frac{b-a}{n+2}$, $\quad x_j = a+(j+1)h \quad j=-1,0,\dots,n+1$

There exists ξ in (a,b) such that the Composite Midpoint rule for $n+2$ subintervals can be written as

$$\int_a^b f(x)\,dx = 2h\sum_{j=0}^{\frac{n}{2}} f(x_{2j}) + \frac{b-a}{6}h^2 f''(\xi)$$

Ex: Use Composite Midpoint rule with $n+2$
#3/203 subintervals to approximate

$$\int_{-2}^{2} x^3 e^x\,dx \qquad n=4$$



$$h = \frac{4}{6} = \frac{2}{3}$$

$$\int_{-2}^{2} x^3 e^x\,dx \approx 2h\left[f(x_0)+f(x_2)+f(x_4)\right]$$

$$\frac{4}{3}\left[-0.6248228459 + 8.99239797258\right]$$

$$= \frac{4}{3} * 8.367575127 = 11.15676684$$

Ex. Determine the values of h and n
#9/204 required to approximate

$$\int_0^2 \frac{1}{x+4}\,dx$$

to within 10^{-5}

a) using composite trapezoidal rule

$$E_n^T(f) = -\frac{b-a}{12} h^2 f''(\xi) = -\frac{1}{6} h^2 f''(\xi)$$

This error can be estimated as follows

$$f(x) = \frac{1}{x+4} \quad f'(x) = \frac{-1}{(x+4)^2} \quad f''(x) = \frac{2}{(x+4)^3}$$

$f''(x)$ is a decreasing function on $[0,2]$.
Thus

$$\max_{[0,2]} |f''(x)| = \frac{2}{4^3} = \frac{2}{4^2 \cdot 4} = \frac{1}{32} = 0.03125$$

$$E_n^T(f) \le \frac{1}{6} h^2\, 0.03125 = 0.0052083\bar{3}\, h^2 < 10^{-5}$$

$$h^2 < \frac{10^{-5}}{0.00521} = \frac{10^{-5}}{5.21 \cdot 10^{-3}} = \frac{10^{-2}}{5.21} = 0.001919$$

$$h < 0.04381$$

$$\frac{b-a}{n} < 0.04381 \quad n > \frac{b-a}{0.04381} = 45.65$$

$$n \ge 46$$

Remark: The errors in composite midpoint and composite trapezoidal are of order $\mathcal{O}(h^2)$ while the error of the composite Simpson's rule is $\mathcal{O}(h^4)$

Ex: Suppose that the approximation to a certain integral using Simpson's rule with $n=10$ is 2.346 and that the exact value is 4. What would the approximation of this integral be using Simpson method with $n=30$.

$$E_{10}^{S}(f) = -\frac{b-a}{180}\left(\frac{b-a}{10}\right)^4 f^{(4)}(\xi)$$

$$E_{30}^{S}(f) = -\frac{b-a}{180}\left(\frac{b-a}{30}\right)^4 f^{(4)}(\bar{\xi})$$

We assume $f^{(4)}(\xi) \approx f^{(4)}(\bar{\xi})$. Thus

$$\frac{E_{10}^{S}(f)}{E_{30}^{S}(f)} \approx \frac{30^4}{10^4} \approx 3^4 = 81$$

$$E_{10}^{S}(f) = 4 - 2.346 = 1.654$$

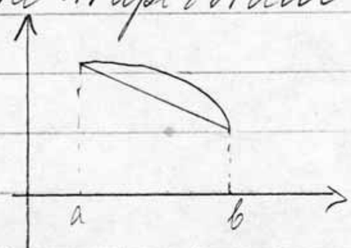
$$E_{10}^{S}(f) = 81\, E_{30}^{S}(f)$$
$$1.654 = 81\, E_{30}^{S}(f)$$

$\Rightarrow E_{30}^{S}(f) = 0.02042$

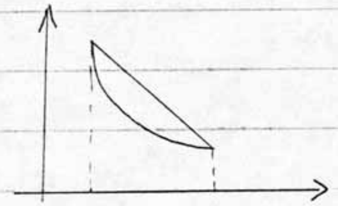
Thus, we can expect that the Simpson rule with 30 subdivisions will give something around 3.97958

Is the approximation an underestimate or an overestimate?

- The Trapezoidal rule:



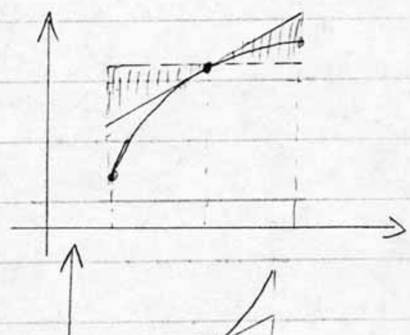
$f(x)$-concave down
TRAP underestimates

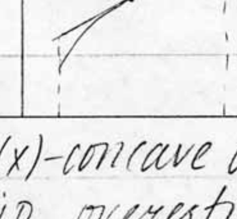


f-concave up
TRAP overestimates

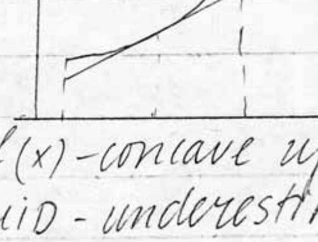
- The Midpoint rule:

Note: AREA under rectangle = AREA under trapezoid





$f(x)$-concave down
MID overestimates



$f(x)$-concave up
MID - underestimates