Review Test 2

Ex: A natural cubic spline is given by

$$S(x)=\begin{cases} S_0(x)=1+\frac{1}{2}(x-1)-(x-1)^3 & 1\le x\le 2 \\ S_1(x)=a+b(x-2)+c(x-2)^2+d(x-2)^3 & 2\le x\le 3 \end{cases}$$

Determine a, b, c, d.

Solution: $S_0(2)=S_1(2)$ $\quad a=1+\frac{1}{2}-1=\frac{1}{2}$ $\quad \boxed{a=\frac{1}{2}}$

$$S'(x)=\begin{cases} S_0'(x)=\frac{1}{2}-3(x-1)^2 & 1\le x\le 2 \\ S_1'(x)=b+2c(x-2)+3d(x-2)^2 & 2\le x\le 3 \end{cases}$$

$S_0'(2)=S_1'(2)$ $\quad \frac{1}{2}-3=b \Rightarrow \boxed{b=-\frac{5}{2}}$

$$S''(x)=\begin{cases} S_0''(x)=-6(x-1) & 1\le x\le 2 \\ S_1''(x)=2c+6d(x-2) & 2\le x\le 3 \end{cases}$$

$S_0''(2)=S_1''(2)$ $\quad -6=2c \Rightarrow \boxed{c=-3}$

$S_0''(1)=0$ - satisfied $\quad S_1''(3)=-6+6d=0$ $\quad \boxed{d=1}$

Ex(a) Use the most accurate three-point formula to determine each missing entry in the table

$3^d/176$

x	$f(x)$	$f'(x)$
2.0	3.68	0.25
2.1	3.69	-0.05
2.2	3.67	-0.35
2.3	3.62	-0.65

$$f'(x_0) = \frac{1}{2h}\left[-3f(x_0) + 4f(x_0+h) - f(x_0+2h)\right] + \frac{h^2}{3} f^{(3)}(\xi_0)$$

$$f'(x_0) = \frac{1}{2h}\left[-f(x_0-h) + f(x_0+h)\right] - \frac{h^2}{6} f^{(3)}(\xi_1)$$

$$f'(x_0) = \frac{1}{2h}\left[f(x_0-2h) - 4f(x_0-h) + 3f(x_0)\right] + \frac{h^2}{3} f^{(3)}(\xi_2)$$

$$f'(2) = \frac{1}{0.2}\left[-3\cdot 3.68 + 4\cdot 3.69 - 3.67\right] =$$

$$= \frac{1}{0.2}\left[-11.04 + 14.76 - 3.67\right]$$

$$= \frac{1}{0.2}(0.05) = \frac{5}{20} = \frac{1}{4}$$

$$f'(2.1) = \frac{1}{0.2}\left[3.67 - 3.68\right] = \frac{1}{0.2}(-0.01) = -\frac{1}{20}$$

$$f'(2.2) = \frac{1}{0.2}\left[3.62 - 3.69\right] = \frac{1}{0.2}(-0.07) = -\frac{7}{20}$$

$$f'(23) = \frac{1}{0.2}\left[3.69 - 4\cdot 3.67 + 3(3.62)\right] =$$

$$= \frac{1}{0.2}\left[-0.13\right] = -0.65$$

Ex. A centered difference is used for computing $f'(-2)$ with $h = 0.2$. Find an upper bound of the error if

$$f(x) = e^{\frac{x}{3}} + x^2$$

Solution:

$$\text{Error} = \frac{h^2}{6} f^{(3)}(\xi_2) \quad \text{where } \xi_2 \text{ in } (-2.2, -1.8)$$

$$f' = \frac{1}{3} e^{\frac{x}{3}} + 2x$$

$$f'' = \frac{1}{9} e^{\frac{x}{3}} + 2$$

$$f''' = \frac{1}{27} e^{\frac{x}{3}} \qquad |f'''(\xi)| \le \frac{1}{27} e^{-\frac{1.8}{3}} = \frac{1}{27} e^{-0.6}$$

$$\text{Error bound} = \frac{1}{5^2} \cdot \frac{1}{6} \cdot \frac{1}{27} e^{-0.6} = \frac{1}{27 \cdot 150} e^{-0.6}$$

Ex. 11/195 skip The quadrature formula

$$\int_{-1}^{1} f(x)\,dx = c_0 f(-1) + c_1 f(0) + c_2 f(1)$$

is exact for all polynomials of degree less or equal to 2. Determine c_0, c_1, c_2

Solution: $\int_{-1}^{1} 1\,dx = 2 \qquad c_0 + c_1 + c_2 = 2$

$$\int_{-1}^{1} x\,dx = 0 \qquad -c_0 + c_2 = 0 \Rightarrow c_0 = c_2$$

$$\int_{-1}^{1} x^2\,dx = \frac{2}{3} \qquad c_0 + c_2 = \frac{2}{3} \qquad 2c_2 = \frac{2}{3} \quad \boxed{c_2 = \frac{1}{3}}$$

From the first equation

$$c_1 = 2 - \frac{2}{3} = \frac{4}{3}$$

Ex. #8/195 The Trapezoidal rule applied to

$$\int_0^2 f(x)\,dx$$

gives value 5, and the Midpoint rule gives value 4. What value does Simpson's rule give?

Solution:

$h=2$ 0   2

$$\int_0^2 f(x)\,dx = \frac{2}{2}\left[f(0)+f(2)\right] = 5$$
$$\Rightarrow f(0)+f(2) = 5$$

$h=1$ 0   1   2

$$\int_0^2 f(x)\,dx = 2(1)\cdot f(1) = 4$$
$$\Rightarrow f(1) = 2$$

$h=1$

Simpson's

$$\int_0^2 f(x)\,dx = \frac{1}{3}\left[f(0)+4f(1)+f(2)\right] =$$
$$= \frac{1}{3}\left[5+4\cdot 2\right] = \frac{1}{3}\left[5+8\right] = \frac{13}{3}$$

Ex. Set up a composite Simpson's method to evaluate the integral

$$\int_0^2 e^{2x} \sin 3x \, dx \quad \text{with } n=8$$

Do not evaluate to final answer.

Solution: 0 0.25 0.5 0.75 1 1.25 1.5 1.75 2

Thus: $h = 0.25$

$$\int_0^2 e^{2x} \sin 3x \, dx = \frac{0.25}{3}\Big[e^{2\cdot 0}\sin 0 + 4e^{2\cdot 0.25}\sin 3(0.25)$$

$$+ 2e^{2\cdot 0.5}\sin 3(0.5) + 4e^{2\cdot 0.75}\sin 3(0.75) + 2e^{2\cdot 1}\sin 3$$

$$+ 4e^{2\cdot 1.25}\sin 3(1.25) + 2e^{2\cdot 1.5}\sin 3(1.5) + 4e^{2\cdot 1.75}\sin 3(1.75)$$

$$+ e^{2\cdot 2}\sin 2\cdot 3\Big]$$

Ex: Set up Gaussian quadrature with $n=3$ to evaluate the integral.

$$\int_0^1 x^2 e^{-x} \, dx$$

Do not simplify your answer to a final value.

$$x = \frac{1}{2}\left[(b-a)t + a + b\right]$$

$$x = \frac{1}{2}(t+1) \qquad dx = \frac{1}{2}dt$$

$$\int_0^1 x^2 e^{-x}\,dx = \frac{1}{8}\int_{-1}^{1}(t+1)^2 e^{-\frac{1}{2}(t+1)}\,dt$$

$$= \frac{1}{8}\left[c_1(t_1+1)^2 e^{-\frac{1}{2}(t_1+1)} + c_2(t_2+1)^2 e^{-\frac{1}{2}(t_2+1)} + c_3(t_3+1)e^{-\frac{1}{2}(t_3+1)}\right]$$

$$= \frac{1}{8}\Big[0.55556(0.7746+1)^2 e^{-\frac{1}{2}(0.7746+1)} + 0.88889\,e^{-\frac{1}{2}}$$

$$+ 0.55556(-0.7746+1)^2 e^{-\frac{1}{2}(-0.7746+1)}\Big]$$

Ex. Determine constants a, b, c, d and e that will produce a quadrature formula

$$\int_{-1}^{1} f(x)\,dx = af(-1) + bf(0) + cf(1) + df'(-1) + ef'(1)$$

that has degree of precision 4.

$f(x) = 1 \qquad f'(x) = 0$

$\int_{-1}^{1} 1\,dx = 2 \qquad a + b + c = 2$

$f(x) = x \quad f'(x) = 1$

$\int_{-1}^{1} x\,dx = 0 \quad a(-1) + c + d + e = 0$

$f(x) = x^2 \quad f'(x) = 2x$

$\int_{-1}^{1} x^2 dx = \frac{2}{3} \quad a + c - 2d + 2e = \frac{2}{3}$

$f(x) = x^3 \quad f'(x) = 3x^2$

$\int_{-1}^{1} x^3 dx = 0 \quad a(-1) + c + 3d + 3e = 0$

$f(x) = x^4 \quad f'(x) = 4x^3$

$\int_{-1}^{1} x^4 dx = \frac{2}{5} \quad a + c - 4d + 4e = \frac{2}{5}$

Thus, we get the system

$$\begin{array}{l} (1) \\ (2) \\ (3) \\ (4) \\ (5) \end{array} \begin{cases} a + b + c = 2 \\ -a + c + d + e = 0 \\ a + c - 2d + 2e = \frac{2}{3} \\ -a + c + 3d + 3e = 0 \\ a + c - 4d + 4e = \frac{2}{5} \end{cases}$$

Subtracting (4) - (2) we get

$$2(d+e)=0 \Rightarrow \boxed{d+e=0} \Rightarrow e=-d$$

Hence, from (2) we get $-a+c=0 \Rightarrow \boxed{c=a}$

Eliminating c & e from (3) & (5)

$$\begin{cases} 2a-2d-2d=\frac{2}{3} \\ 2a-4d-4d=\frac{2}{5} \end{cases}$$

$$\begin{cases} 2a-4d=\frac{2}{3} \\ 2a-8d=\frac{2}{5} \end{cases} \qquad \begin{cases} a-2d=\frac{1}{3} \\ a-4d=\frac{1}{5} \end{cases}$$

$$-2d-(-4d)=\frac{1}{3}-\frac{1}{5}$$

$$2d=\frac{2}{15} \quad \boxed{d=\frac{1}{15}}$$

$$a=\frac{1}{3}+\frac{2}{15}=\frac{7}{15} \quad c=\frac{7}{15} \quad e=-\frac{1}{15}$$

From (1) $\quad \frac{14}{15}+b=2 \quad b=2-\frac{14}{15}=\frac{16}{15}$

Ex. Determine the values of n and h required to approximate

$$\int_0^1 e^x\,dx$$

within 10^{-6} using Composite Simpson's rule.

<u>Solution</u>: The error of composite Simpson's rule is

$$E^S(f) = -\frac{b-a}{180}\,h^4 f^{IV}(\xi)$$

where ξ in (a,b).

Here $a=0$, $b=1$.

$$f(x) = e^x$$

$$|f^{IV}(x)| = |e^x| \le e$$

$$|E^S(f)| = \frac{1}{180}\,h^4 e \le 10^{-6}$$

We solve for h

$$h^4 \le \frac{180}{e}\cdot 10^{-6} = 6.62183 * 10^{-5}$$

$$h \le 0.0902078861$$