PROBLEM 7.1:

$$x_1[n] = \delta[n] \quad \Longrightarrow \quad X_1(z) = \sum_{n=-\infty}^{\infty} \delta[n] z^{-n} = 1$$

Delay Prop.

$$x_2[n] = \delta[n-1] = x_1[n-1]$$

$$\Longrightarrow \quad X_2(z) = z^{-1} X_1(z) = z^{-1}$$

$$x_3[n] = \delta[n-7] = x_1[n-7]$$

$$\Longrightarrow \quad X_3(z) = z^{-7} X_1(z) = z^{-7}$$

$$x_4[n] = 2\delta[n] - 3\delta[n-1] + 4\delta[n-3]$$

$$= 2x_1[n] - 3x_1[n-1] + 4x_1[n-3]$$

$$X_4(z) = 2X_1(z) - 3z^{-1}X_1(z) + 4z^{-3}X_1(z)$$

$$= 2 - 3z^{-1} + 4z^{-3}$$

McClellan, Schafer and Yoder, *Signal Processing First*, ISBN 0-13-065562-7.
Pearson Prentice Hall, Inc. Upper Saddle River, NJ 07458. © 2003

PROBLEM 7.2:

$$y[n] = x[n] - x[n-1]$$

DELAY PROPERTY

$$Y(z) = X(z) - z^{-1}X(z)$$

$$= (1 - z^{-1})X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{(1-z^{-1})X(z)}{X(z)} = 1 - z^{-1}$$

This difference equation is the definition of the "first (backward) difference" operation.

McClellan, Schafer and Yoder, *Signal Processing First*, ISBN 0-13-065562-7.
Pearson Prentice Hall, Inc. Upper Saddle River, NJ 07458. © 2003

PROBLEM 7.3:

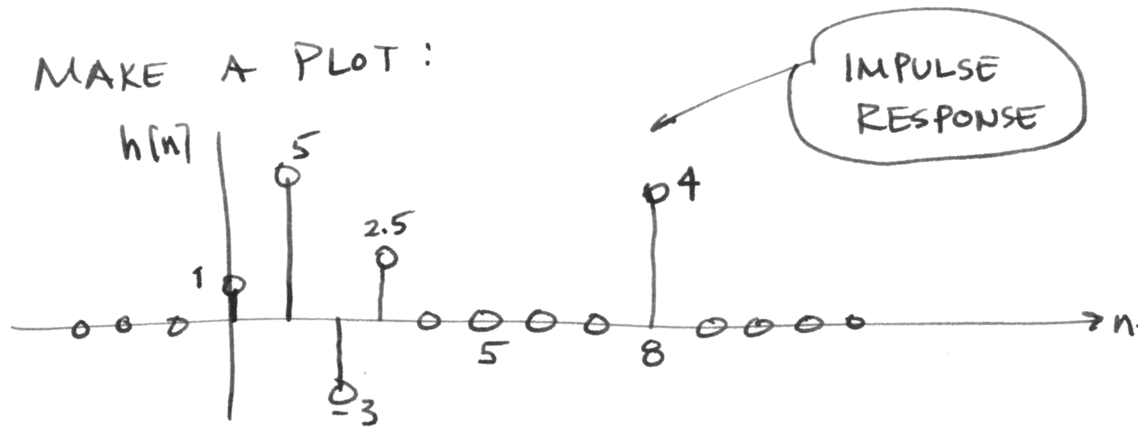
(a) $y[n] = x[n] + 5x[n-1] - 3x[n-2] + \frac{5}{2}x[n-3] + 4x[n-8]$

$$H(z) = 1 + 5z^{-1} - 3z^{-2} + \frac{5}{2}z^{-3} + 4z^{-8}$$

(b) when $x[n] = \delta[n]$, you can substitute.

$$h[n] = \delta[n] + 5\delta[n-1] - 3\delta[n-2] + \frac{5}{2}\delta[n-3] + 4\delta[n-8]$$

MAKE A PLOT:



NOTE:

The difference equation can be written as:

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

Then the impulse response will just take on the values given by the $\{b_k\}$

$\therefore$ $h[0] = b_0$, $h[1] = b_1$, $h[2] = b_2$, ... etc.

McClellan, Schafer and Yoder, *Signal Processing First*, ISBN 0-13-065562-7.
Pearson Prentice Hall, Inc. Upper Saddle River, NJ 07458. © 2003

PROBLEM 7.4:

(a) use filter coeffs: $H(z) = \frac{1}{3} + \frac{1}{3}z^{-1} + \frac{1}{3}z^{-2}$

(b) Use positive powers to extract poles and zeros

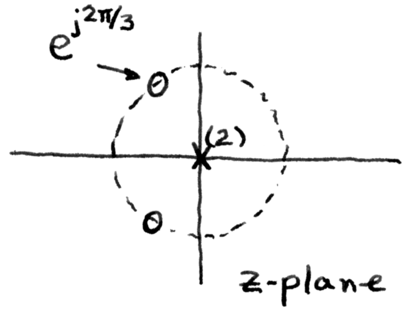
$$H(z) = \frac{1}{z^2}\left(\frac{1}{3}z^2 + \frac{1}{3}z + \frac{1}{3}\right)$$

TWO POLES AT $z=0$

zeros at

$$z = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$$

zeros: $1e^{\pm j2\pi/3}$



(c) $\mathcal{H}(\hat{\omega}) = H(e^{j\hat{\omega}}) = H(z)\big|_{z=e^{j\hat{\omega}}}$

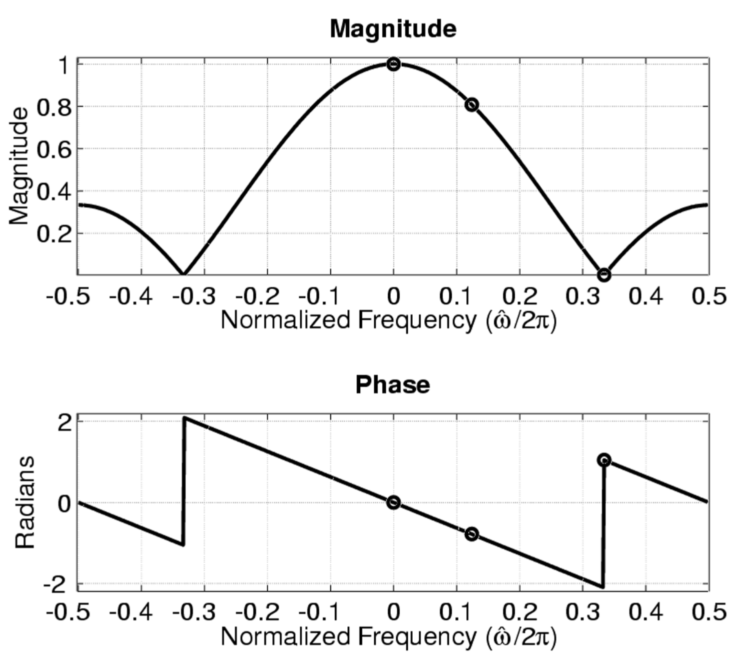
$$= \frac{1}{3} + \frac{1}{3}e^{-j\hat{\omega}} + \frac{1}{3}e^{-j2\hat{\omega}} = \frac{1}{3}e^{-j\hat{\omega}}\left(e^{j\hat{\omega}} + 1 + e^{-j\hat{\omega}}\right)$$

$$= e^{-j\hat{\omega}}\left(\frac{1+2\cos\hat{\omega}}{3}\right)$$

ANOTHER FORMULA:

$$\mathcal{H}(\hat{\omega}) = e^{-j\hat{\omega}}\left(\frac{\sin(3\hat{\omega}/2)}{3\sin(\hat{\omega}/2)}\right)$$

(d) use MATLAB



McClellan, Schafer and Yoder, *Signal Processing First*, ISBN 0-13-065562-7.
Pearson Prentice Hall, Inc. Upper Saddle River, NJ 07458.

PROBLEM 7.4 (more):

(e) Use Linearity & Frequency response at $\hat{\omega}=0$, $\hat{\omega}=\pi/4$ and $\hat{\omega}=2\pi/3$. These are marked on the plots of the frequency response.

$$y[n] = 4\mathcal{H}(0) + |\mathcal{H}(\pi/4)| \cos\left(\frac{\pi}{4}n - \frac{\pi}{4} + \angle\mathcal{H}(\pi/4)\right)$$
$$-3\underbrace{|\mathcal{H}(2\pi/3)|}_{=0} \cos\left(\frac{2\pi}{3}n + \angle\mathcal{H}(2\pi/3)\right)$$

$$\mathcal{H}(0) = \tfrac{1}{3}+\tfrac{1}{3}+\tfrac{1}{3} = 1$$

$$\mathcal{H}(\pi/4) = e^{-j\pi/4}(1+2\sqrt{2}/2)/3 = \frac{1+\sqrt{2}}{3}e^{-j\pi/4} = 0.8047e^{-j\pi/4}$$

$\mathcal{H}(2\pi/3) = 0$ because $H(z)=0$ at $z = e^{\pm j2\pi/3}$

$$\therefore\ y[n] = 4 + 0.8047\cos\left(\frac{\pi}{4}n - \frac{\pi}{2}\right)$$

McClellan, Schafer and Yoder, *Signal Processing First*, ISBN 0-13-065562-7.
Pearson Prentice Hall, Inc. Upper Saddle River, NJ 07458. © 2003

PROBLEM 7.5:

(a)

$$H(z) = (1-z^{-1})\underbrace{(1-jz^{-1})(1+jz^{-1})}_{1+z^{-2}}\underbrace{(1-0.9e^{j\pi/3}z^{-1})(1-0.9e^{-j\pi/3}z^{-1})}_{1-\underbrace{1.8\cos(\pi/3)}_{0.9}z^{-1}+0.81z^{-2}}$$

$$H(z) = (1-z^{-1}+z^{-2}-z^{-3})(1-0.9z^{-1}+0.81z^{-2})$$

$$= 1-1.9z^{-1}+2.71z^{-2}-2.71z^{-3}+1.71z^{-4}-0.81z^{-5}$$

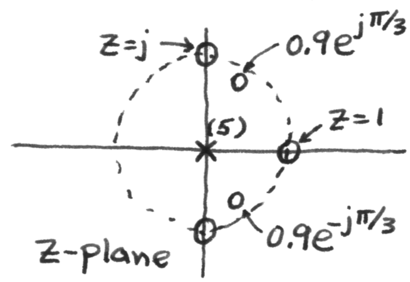
Use polynomial coeffs as filter coeffs:

$$y[n] = x[n]-1.9x[n-1]+2.71x[n-2]-2.71x[n-3]+1.71x[n-4]-0.81x[n-5]$$

(b) $H(z) = \frac{1}{z^5}(z-1)(z-j)(z-(-j))(z-0.9e^{j\pi/3})(z-0.9e^{-j\pi/3})$

FIVE POLES AT $z=0$

THE FACTORED FORM GIVES ALL THE ZEROS



(c) The zeros on the unit circle will cause nulling of $x[n] = Ae^{j\varphi}e^{j\hat{\omega}n}$

$z = 1 = e^{j0}$ so $\hat{\omega} = 0$ is nulled

$z = j = e^{j\pi/2}$ so $e^{j\frac{\pi}{2}n}$ is nulled

$z = -j = e^{-j\pi/2}$ so $e^{-j\frac{\pi}{2}n}$ is nulled.

PROBLEM 7.6:

(a) $Y_1(z) = H_1(z)X(z)$

$$Y(z) = H_2(z)Y_1(z) = H_2(z)\big(H_1(z)X(z)\big)$$
$$= \underbrace{\big(H_2(z)H_1(z)\big)}_{H(z)}X(z) \quad \text{because } H(z) = \frac{Y(z)}{X(z)}$$

(b) Since $H_2(z)H_1(z) = H_1(z)H_2(z)$ because $H_1(z)$ and $H_2(z)$ are scalar functions.

$$\Rightarrow Y(z) = H_1(z)\underbrace{H_2(z)X(z)}_{\text{means that } H_2(z) \text{ is applied first}}$$

(c) $H_1(z) = \frac{1}{3}(1+z^{-1}+z^{-2})$ by using the filter coeffs.

$$H(z) = H_2(z)H_1(z)$$
$$= \tfrac{1}{3}(1+z^{-1}+z^{-2})\cdot\tfrac{1}{3}(1+z^{-1}+z^{-2})$$
$$= \tfrac{1}{9}(1+2z^{-1}+3z^{-2}+2z^{-3}+z^{-4})$$

(d) Convert to difference equation (i.e., filter coeffs)

$$y[n] = \tfrac{1}{9}\big(x[n]+2x[n-1]+3x[n-2]+2x[n-3]+x[n-4]\big)$$

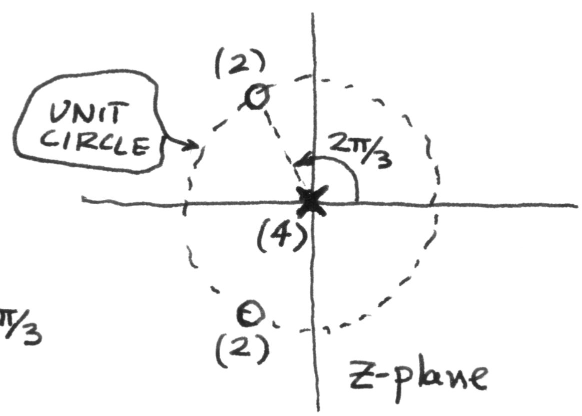
(e) Find the poles & zeros of $H_2(z)$, then "double" them because $H_1(z) = H_2(z)$.

$$H_2(z) = \tfrac{1}{3}z^{-2}(z^2+z+1)$$

$\frac{1}{z^2}$ contributes two poles at $z=0$

zeros are:

$$\frac{-1\pm\sqrt{1-4}}{2} = -\tfrac{1}{2}\pm j\tfrac{\sqrt{3}}{2} = e^{\pm j2\pi/3}$$



PROBLEM 7.6 (more):

(f)

$$H(e^{j\hat{\omega}}) = H_1(e^{j\hat{\omega}})H_2(e^{j\hat{\omega}})$$
$$= \frac{1}{9}\left(1 + e^{-j\hat{\omega}} + e^{-j2\hat{\omega}}\right)^2$$
$$= \frac{1}{9}e^{-j2\hat{\omega}}\left(e^{j\hat{\omega}} + 1 + e^{-j\hat{\omega}}\right)^2$$
$$= \frac{1}{9}e^{-j2\hat{\omega}}\left(1 + 2\cos(\hat{\omega})\right)^2$$

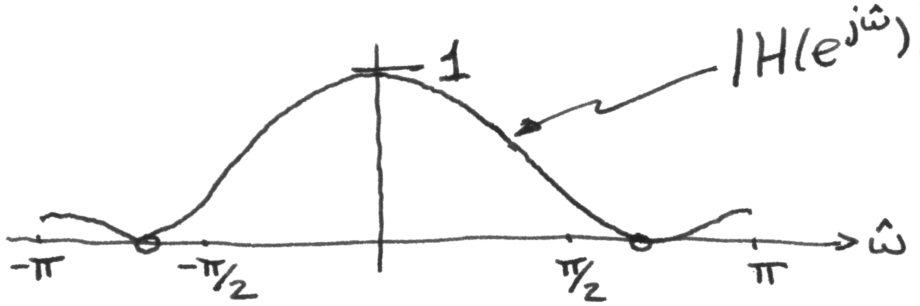
$$|H(e^{j\hat{\omega}})| = \frac{1}{9}\left(1 + 2\cos(\hat{\omega})\right)^2$$

At $\hat{\omega} = 0$, $|H| = \frac{1}{9}(3)^2 = 1$

At $\hat{\omega} = \pi/2$, $|H| = \frac{1}{9}(1)^2 = 1/9$

At $\hat{\omega} = 2\pi/3$, $|H| = 0$ because there is a zero on the unit circle.

At $\hat{\omega} = \pi$, $|H| = \frac{1}{9}(1-2)^2 = 1/9$



McClellan, Schafer and Yoder, *Signal Processing First*, ISBN 0-13-065562-7.
Pearson Prentice Hall, Inc. Upper Saddle River, NJ 07458.

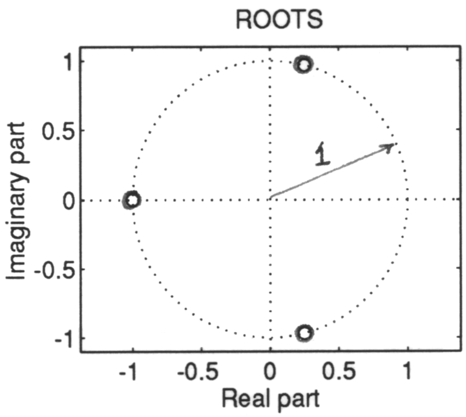
PROBLEM 7.7:

$$P(z) = 1 + \frac{1}{2}z^{-1} + \frac{1}{2}z^{-2} + z^{-3}$$

NOTE $P(-1) = 1 - \frac{1}{2} + \frac{1}{2} - 1 = 0 \implies$ ROOT @ $z = -1$

$$\implies P(z) = (1 + z^{-1})\left(1 - \frac{1}{2}z^{-1} + z^{-2}\right)$$

USE QUADRATIC FORMULA ON THIS PART



$$\frac{\frac{1}{2} \pm \sqrt{1/4 - 4}}{2}$$

$$= \underbrace{\frac{1}{4} \pm j\frac{\sqrt{15}}{4}}$$

MAG OF THESE ROOTS IS EXACTLY ONE.

McClellan, Schafer and Yoder, *Signal Processing First*, ISBN 0-13-065562-7.
Pearson Prentice Hall, Inc. Upper Saddle River, NJ 07458. © 2003

PROBLEM 7.8:

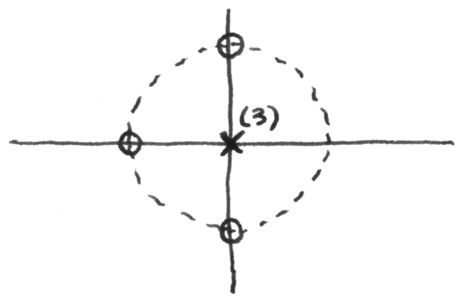
(a) $h[n] = \frac{1}{4}\{\delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3]\}$

(b) $H(z) = \frac{1}{4}(1 + z^{-1} + z^{-2} + z^{-3})$ by using $h[n]$.

(c) Poles and zeros:

$$H(z) = \frac{1}{4}\,\frac{z^3+z^2+z+1}{z^3}$$

3 Poles at $z=0$



$$z^3+z^2+z+1 = \frac{z^4-1}{z-1}$$

zeros at $z=\pm j$ & $z=-1$

(d) $H(z) = \frac{1}{4}\,\frac{1-z^{-4}}{1-z^{-1}} = \frac{1}{4}(1+z^{-1}+z^{-2}+z^{-3})$

$$H(e^{j\hat{\omega}}) = \frac{1}{4}\,\frac{1-e^{-j4\hat{\omega}}}{1-e^{-j\hat{\omega}}} = \frac{1}{4}\,\frac{e^{-j2\hat{\omega}}(e^{j2\hat{\omega}}-e^{-j2\hat{\omega}})}{e^{-j\hat{\omega}/2}(e^{+j\hat{\omega}/2}-e^{-j\hat{\omega}/2})}$$

$$= \frac{1}{4}e^{-j3\hat{\omega}/2}\,\frac{\sin(2\hat{\omega})}{\sin(\frac{1}{2}\hat{\omega})}$$

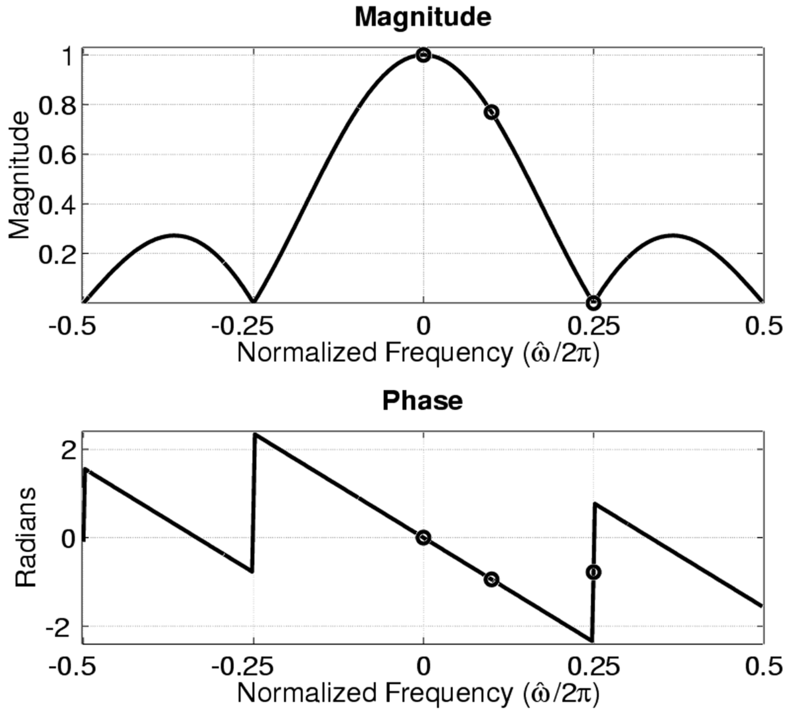
At $\hat{\omega}=0$, $H(e^{j\hat{\omega}}) = \frac{1}{4}e^{j0}\cdot 4 = 1$

At $\hat{\omega} = \pi/2, \pi, -\pi/2$, $H(e^{j\hat{\omega}}) = 0$ because $\sin(2\hat{\omega})=0$.

McClellan, Schafer and Yoder, *Signal Processing First*, ISBN 0-13-065562-7.
Pearson Prentice Hall, Inc. Upper Saddle River, NJ 07458.

PROBLEM 7.8 (more):

(e) use MATLAB



(f) Evaluate $H(e^{j\hat{\omega}})$ at $\hat{\omega}=0$, $\hat{\omega}=0.2\pi$ and $\hat{\omega}=0.5\pi$.

These are marked on the frequency response plots

$H(e^{j0})=1 \qquad H(e^{j0.2\pi})=0.771e^{-j0.3\pi} \qquad H(e^{j0.5\pi})=0$

$\Rightarrow y[n] = 5 + 4(0.771)\cos(0.2\pi n - 0.3\pi) + 0$

$= 5 + 3.084\cos(0.2\pi n - 0.3\pi)$

ANGLE = -54° or -0.94 rads

McClellan, Schafer and Yoder, *Signal Processing First*, ISBN 0-13-065562-7.
Pearson Prentice Hall, Inc. Upper Saddle River, NJ 07458. © 2003

PROBLEM 7.9:

(a) A 4-point moving average is

$$y[n] = \frac{1}{4}\left(x[n] + x[n-1] + x[n-2] + x[n-3]\right)$$

$$H_1(z) = H_2(z) = \frac{1}{4} + \frac{1}{4}z^{-1} + \frac{1}{4}z^{-2} + \frac{1}{4}z^{-3}$$

$$H(z) = H_1(z)H_2(z) = \frac{1}{16}\left(1 + z^{-1} + z^{-2} + z^{-3}\right)^2$$

(b) Find the poles and zeros of $H_1(z)$, then "double" them. Switch to positive powers of z.

$$H_1(z) = \frac{z^3 + z^2 + z + 1}{4z^3}$$

Numerator factors:

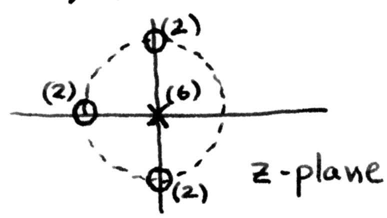
$$(z+1)(z^2+1)$$
$$= (z+1)(z+j)(z-j)$$

3 poles at $z=0$

$\Rightarrow$ Zeros at $z = -1, -j, +j$

(c) For the freq. response

$$H(e^{j\hat{\omega}}) = H(z)\big|_{z=e^{j\hat{\omega}}}$$



$$H(e^{j\hat{\omega}}) = \frac{1}{16}\left(\underbrace{1 + e^{-j\hat{\omega}} + e^{-j2\hat{\omega}} + e^{-j3\hat{\omega}}}\right)^2$$

This can be reduced to a Dirichlet form.

$$H(e^{j\hat{\omega}}) = \frac{1}{16}\left(\frac{\sin(4\hat{\omega}/2)}{\sin(\hat{\omega}/2)}\,e^{-j3\hat{\omega}/2}\right)^2$$

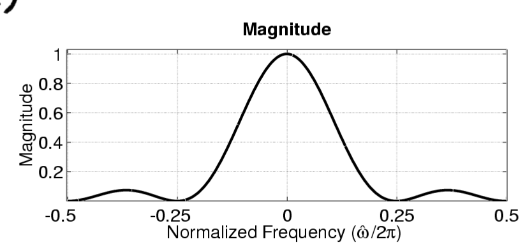
$$= e^{-j3\hat{\omega}}\left(\frac{\sin(2\hat{\omega})}{4\sin(\hat{\omega}/2)}\right)^2$$

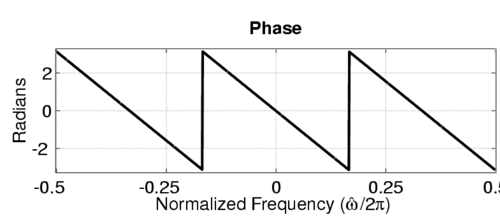
phase

Numerator is zero for $2\hat{\omega} = \pi k$
$\Rightarrow \hat{\omega} = \pi k/2$

At $\hat{\omega}=0$, denominator is also zero

(d)





McClellan, Schafer and Yoder, *Signal Processing First*, ISBN 0-13-065562-7.
Pearson Prentice Hall, Inc. Upper Saddle River, NJ 07458. © 2003

PROBLEM 7.9 (more):

(e)

$$H(z)=\frac{1}{16}\left(1+z^{-1}+z^{-2}+z^{-3}\right)^2$$

$$=\frac{1}{16}\left(1+2z^{-1}+3z^{-2}+4z^{-3}+3z^{-4}+2z^{-5}+z^{-6}\right)$$

Invert term by term

$$h[n]=\frac{1}{16}\delta[n]+\frac{1}{8}\delta[n-1]+\frac{3}{16}\delta[n-2]+\frac{1}{4}\delta[n-3]+\frac{3}{16}\delta[n-4]$$

$$+\frac{1}{8}\delta[n-5]+\frac{1}{16}\delta[n-6]$$

McClellan, Schafer and Yoder, *Signal Processing First*, ISBN 0-13-065562-7.
Pearson Prentice Hall, Inc. Upper Saddle River, NJ 07458.

PROBLEM 7.10:

(a) Convert $H(z)$ to a difference equation:

$$y[n] = x[n] - 3x[n-2] + 2x[n-3] + 4x[n-6]$$

The most delay is 6 samples, so the term $4\delta[n-4]$ in $x[n]$ is delayed to $16\delta[n-10]$.

The least amount of delay is $2\delta[n]$ experiencing no delay. Thus the output starts at $n=0$ and ends at $n=10$.

$\Rightarrow$ $y[n]=0$ for $n<0$ & $n>10$

$N_1=0$ and $N_2=10$.

(b) $X(z) = 2 + z^{-1} - 2z^{-2} + 4z^{-4}$

$$\begin{aligned} Y(z) &= H(z)X(z) \\ &= (1-3z^{-2}+2z^{-3}+4z^{-6})(2+z^{-1}-2z^{-2}+4z^{-4}) \\ &= 2+z^{-1}-2z^{-2}+4z^{-4}-6z^{-2}-3z^{-3}+6z^{-4}-12z^{-6} \\ &\quad +4z^{-3}+2z^{-4}-4z^{-5}+8z^{-7}+8z^{-6}+4z^{-7}-8z^{-8}+16z^{-10} \end{aligned}$$

Combine terms with common exponents

$$\begin{aligned} Y(z) = {} & 2+z^{-1}-8z^{-2}+z^{-3}+12z^{-4}-4z^{-5}-4z^{-6}+12z^{-7} \\ & -8z^{-8}+16z^{-10} \end{aligned}$$

Invert:

$$\begin{aligned} y[n] = {} & 2\delta[n]+\delta[n-1]-8\delta[n-2]+\delta[n-3]+12\delta[n-4] \\ & -4\delta[n-5]-4\delta[n-6]+12\delta[n-7]-8\delta[n-8]+16\delta[n-10] \end{aligned}$$

PROBLEM 7.11:

```
omeg = pi/6;
nn = [ 0:29 ];
xn = sin(omeg*nn);
bb = [ 1  0  0  1 ];
aa = [ 1 ];
yn = filter( bb, aa, xn );        %<--- alternate form:   yn = conv( bb, xn )
```

(a) $H(z) = 1 + 0z^{-1} + 0z^{-2} + 1z^{-3}$

$= 1 + z^{-3} \longrightarrow$ ROOTS: $\{-1, e^{j\pi/3}, e^{-j\pi/3}\}$

(b) Determine a formula for $y[n]$, the signal contained in the vector **yn**.

$x[n] = \sin(\pi n/6) = \cos(\pi n/6 - \pi/2)$

Use $H(e^{j\hat{\omega}})$ at $\hat{\omega} = \pi/6$

$H(e^{j\hat{\omega}}) = H(z)\big|_{z=e^{j\hat{\omega}}} = 1 + e^{-j3\hat{\omega}}$

AT $\hat{\omega} = \pi/6$ $\quad H(e^{j\hat{\omega}}) = 1 + e^{-j3\pi/6} = 1 + e^{-j\pi/2} = 1 - j$

$= \sqrt{2}\, e^{-j\pi/4}$

$\therefore\ y[n] = |H(e^{j\pi/6})| \cos(\pi n/6 - \pi/2 + \angle H(e^{j\pi/6}))$

$= \sqrt{2} \cos(\pi n/6 - \pi/2 - \pi/4)$

$= \sqrt{2} \cos(\pi n/6 - 3\pi/4)$

(c) Give a value of **omeg** such that the output is guaranteed to be zero, for $n \geq 3$.

ALL ZEROS LIE ON UNIT CIRCLE IN Z-plane

$\Rightarrow H(e^{j\hat{\omega}}) = 0$ for $\hat{\omega} = \pi/3, -\pi/3$ or π

when $H(e^{j\hat{\omega}}) = 0$ the output will be zero for $n \geq 3$

$\therefore$ omeg $= \pi/3$ or omeg $= \pi$

McClellan, Schafer and Yoder, *Signal Processing First*, ISBN 0-13-065562-7.
Pearson Prentice Hall, Inc. Upper Saddle River, NJ 07458. © 2003

PROBLEM 7.12:

(a)

MULTIPLY OUTER FACTORS

$$H(z) = (1-z^{-1})(1+z^{-2})(1+z^{-1})$$
$$= (1-z^{-2})(1+z^{-2}) = 1-z^{-4}$$

$\therefore \quad y[n] = x[n] - x[n-4]$

(b) $H(e^{j\hat{\omega}}) = H(z)\big|_{z=e^{j\hat{\omega}}} = 1 - e^{-j4\hat{\omega}}$

(c) $H(e^{j\hat{\omega}}) = e^{-j2\hat{\omega}}\left(e^{+j2\hat{\omega}} - e^{-j2\hat{\omega}}\right)$
$$= 2j\, e^{-j2\hat{\omega}} \sin 2\hat{\omega} = (2\sin 2\hat{\omega})\, e^{j(\pi/2 - 2\hat{\omega})}.$$

MAG: $\quad 2\sin 2\hat{\omega}$

ALTHOUGH THIS HAS A SIGN CHANGE FOR $\hat{\omega}<0$

PHASE: $\quad \pi/2 - 2\hat{\omega}$

(d) BLOCK WHEN $H(e^{j\hat{\omega}}) = 0$

$\therefore$ SOLVE $\quad 2\sin 2\hat{\omega} = 0$

$\Longrightarrow \quad \hat{\omega} = 0, \pi/2, \pi, -\pi/2$

(e) Need $H(e^{j\pi/3})$ because that is the frequency of the input.

$$H(e^{j\pi/3}) = \left(2\sin \tfrac{2\pi}{3}\right) e^{j(\pi/2 - 2\pi/3)}$$
$$= 2\left(-\tfrac{\sqrt{3}}{2}\right) e^{j(3\pi/6 - 4\pi/6)}$$
$$= -\sqrt{3}e^{-j\pi/6} = \sqrt{3}e^{j\pi}e^{-j\pi/6} = \sqrt{3}\,e^{j5\pi/6}$$

$\therefore$ OUTPUT IS: $\quad y[n] = \sqrt{3}\cos\left(\frac{\pi n}{3} + \frac{5\pi}{6}\right)$

McClellan, Schafer and Yoder, *Signal Processing First*, ISBN 0-13-065562-7.
Pearson Prentice Hall, Inc. Upper Saddle River, NJ 07458. © 2003

PROBLEM 7.13:

$$y[n] = x[n] - \sqrt{2}\, x[n-1] + x[n-2] \qquad (7.6.7)$$

$$x[n] = A\cos(\tfrac{\pi}{4}n + \varphi) = \Re e\{ A e^{j\varphi} e^{j\pi n/4} \}$$

$$= \tfrac{1}{2} A e^{j\varphi} e^{j\pi n/4} + \tfrac{1}{2} A e^{-j\varphi} e^{-j\pi n/4}$$

Plug into the difference equation:

$$y[n] = \tfrac{1}{2} A e^{j\varphi} e^{j\pi n/4} + \tfrac{1}{2} A e^{-j\varphi} e^{-j\pi n/4}$$
$$- \tfrac{\sqrt{2}}{2} A e^{j\varphi} e^{j\pi(n-1)/4} - \tfrac{\sqrt{2}}{2} A e^{-j\varphi} e^{-j\pi(n-1)/4}$$
$$+ \tfrac{1}{2} A e^{j\varphi} e^{j\pi(n-2)/4} + \tfrac{A}{2} e^{-j\varphi} e^{-j\pi(n-2)/4}$$

Collect the common terms:

$$y[n] = \tfrac{1}{2} A e^{j\varphi} e^{j\pi n/4} \left(1 - \sqrt{2}\, e^{-j\pi/4} + e^{-j2\pi/4}\right)$$
$$+ \tfrac{1}{2} e^{-j\varphi} e^{-j\pi n/4} \left(1 - \sqrt{2}\, e^{+j\pi/4} + e^{j2\pi/4}\right)$$

The terms in parentheses are zero:

$$1 - \sqrt{2}\, e^{-j\pi/4} + e^{-j\pi/2} = 1 - (1-j) + (-j) = 1 - 1 + j - j = 0$$
$$1 - \sqrt{2}\, e^{+j\pi/4} + e^{j\pi/2} = 1 - (1+j) + j = 0$$

A quicker solution uses $H(z)$

$$H(z) = \frac{Y(z)}{X(z)} = 1 - \sqrt{2}\, z^{-1} + z^{-2}$$

The zeros of $H(z)$ are $z = \dfrac{\sqrt{2} \pm \sqrt{2-4}}{2}$

$$z = \frac{\sqrt{2}}{2} \pm j\frac{\sqrt{2}}{2} = e^{\pm j\pi/4}$$ ← These zeros are on the unit circle, so the signals $e^{\pm j\pi n/4}$ are nulled.

PROBLEM 7.14:

$$H(z) = 1 - 2z^{-2} - 4z^{-4}$$

$$h[n] = \delta[n] - 2\delta[n-2] - 4\delta[n-4]$$

$$x[n] = 20e^{j0n} + 20\cos\left(\tfrac{\pi}{2}n + \tfrac{\pi}{4}\right) - 20\delta[n]$$

$$H(e^{j0})\cdot 20 \qquad \text{Need } H(e^{j\pi/2}) \qquad -20\,h[n]$$

$$H(e^{j\hat{\omega}}) = 1 - 2e^{-j2\hat{\omega}} - 4e^{-j4\hat{\omega}}$$

$$H(e^{j0}) = 1 - 2 - 4 = -5$$

$$H(e^{j\pi/2}) = 1 - 2e^{-j\pi} - 4e^{-j2\pi}$$

$$= 1 + 2 - 4 = -1$$

$$\therefore\; y[n] = -100 - 20\cos\left(\tfrac{\pi}{2}n + \tfrac{\pi}{4}\right) - 20\delta[n] + 40\delta[n-2] + 80\delta[n-4]$$

McClellan, Schafer and Yoder, *Signal Processing First*, ISBN 0-13-065562-7.
Pearson Prentice Hall, Inc. Upper Saddle River, NJ 07458. © 2003

PROBLEM 7.15:

$$x(t) = 4 + \cos(250\pi t - \pi/4) - 3\cos\left(\frac{2000\pi}{3} t\right)$$

with $f_s = 1000$

$$x[n] = x(t)\Big|_{t = n/f_s} = 4 + \cos\left(\frac{\pi n}{4} - \frac{\pi}{4}\right) - 3\cos\left(\frac{2\pi}{3} n\right).$$

Now, run $x[n]$ through the filter $H(z)$.

To do so, we need frequency response at $\hat{\omega} = 0,\ \pi/4,\ 2\pi/3$ $\qquad H(e^{j\hat{\omega}}) = \dfrac{1 + e^{-j\hat{\omega}} + e^{-j2\hat{\omega}}}{3}$

$$H(e^{j0}) = \frac{1+1+1}{3} = 1$$

$$H(e^{j\pi/4}) = \frac{1}{3}\left(1 + e^{-j\pi/4} + e^{-j\pi/2}\right) = \frac{1}{3}\left(1 + \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} - j\right)$$

$$= \frac{2+\sqrt{2}}{6} - j\,\frac{2+\sqrt{2}}{6} = 0.569 - j0.569 = .8047\, e^{-j\pi/4}$$

$$H(e^{j2\pi/3}) = \frac{1}{3}\left(1 + e^{-j2\pi/3} + e^{-j4\pi/3}\right) = 0$$

So, the output of the digital filter is:

$$y[n] = 4 + 0.8047\cos\left(\frac{\pi}{4} n - \frac{\pi}{2}\right) + 0$$

Now convert back to analog

$$n \longrightarrow f_s t = 1000t$$

$$y(t) = 4 + 0.8047\cos(250\pi t - \pi/2)$$

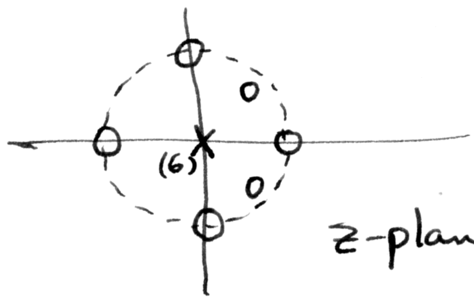
$$\text{or} \quad = 4 + 0.8047\sin(250\pi t)$$

McClellan, Schafer and Yoder, *Signal Processing First*, ISBN 0-13-065562-7.
Pearson Prentice Hall, Inc. Upper Saddle River, NJ 07458. © 2003

PROBLEM 7.16:

(a) $H(z)$ has 6 zeros & 6 poles at $z=0$

The zeros are: $z = \pm 1, \pm j, 0.8e^{\pm j\pi/4}$



z-plane

(b) $w[n] = x[n] - x[n-4]$

$$\Rightarrow \quad H_1(z) = 1 - z^{-4} = (1+z^{-2})(1-z^{-2}).$$

To get $H_2(z)$ divide:

$$H_2(z) = \frac{H(z)}{H_1(z)} = (1 - 0.8e^{-j\pi/4}z^{-1}) \cdot (1 - 0.8e^{+j\pi/4}z^{-1}).$$

$$H_2(z) = 1 - 1.6\cos\pi/4 \; z^{-1} + .64z^{-2}$$

(c) $$y[n] = x[n] - \underbrace{(0.8\sqrt{2})}_{=1.1314}x[n-1] + 0.64x[n-2].$$

PROBLEM 7.17:

$$H(z) = b_0(1 - z^{-4}) + b_1(z^{-1} - z^{-3})$$
$$= z^{-2} b_0 (z^2 - z^{-2}) + z^{-2} b_1 (z - z^{-1})$$

(a) $H(e^{j\hat{\omega}}) = H(z)\big|_{z=e^{j\hat{\omega}}}$

$$= e^{-j2\hat{\omega}} b_0 (\underbrace{e^{j2\hat{\omega}} - e^{-j2\hat{\omega}}}_{2j\sin(2\hat{\omega})}) + e^{-j2\hat{\omega}} b_1 (\underbrace{e^{j\hat{\omega}} - e^{-j\hat{\omega}}}_{2j\sin\hat{\omega}})$$

$$H(e^{j\hat{\omega}}) = \left[2b_0 \sin(2\hat{\omega}) + 2b_1 \sin(\hat{\omega})\right] e^{j(\pi/2 - 2\hat{\omega})} \qquad j = e^{j\pi/2}$$

(b) $H(1/z) = b_0(1 - z^4) + b_1(z - z^3)$

$$= z^2 b_0 (z^{-2} - z^2) + z^2 b_1 (z^{-1} - z)$$
$$= -z^4 \Big[\underbrace{z^{-2} b_0 (z^2 - z^{-2}) + z^{-2} b_1 (z - z^{-1})}_{H(z)} \Big]$$

$$\therefore H(1/z) = -z^4 H(z)$$

(c) Generalize when $b_k = -b_{M-k}$ for $k = 0, 1, \ldots M$

When M is even

$$b_{M/2} = -b_{M-M/2} = -b_{M/2} \Rightarrow b_{M/2} = 0$$

Then we can group terms as in part (a)

$$H(z) = \sum_{k=0}^{M} b_k z^{-k}$$

$$= z^{-M/2} b_0 (z^{M/2} - z^{-M/2}) + z^{-M/2} b_1 (z^{\frac{M}{2}-1} - z^{-\frac{M}{2}+1}) + \ldots$$

Each term in parentheses will become a sine function

McClellan, Schafer and Yoder, *Signal Processing First*, ISBN 0-13-065562-7.
Pearson Prentice Hall, Inc. Upper Saddle River, NJ 07458. © 2003

$$H(e^{j\hat{\omega}}) = e^{-jM\hat{\omega}/2} \sum_{k=0}^{\frac{M}{2}-1} 2j\, b_k \sin\left(\left(\tfrac{M}{2}-k\right)\hat{\omega}\right)$$

$j = e^{j\pi/2}$

For M = odd integer:

$\frac{M}{2}$ is not an integer, so there is no $b_{\frac{M}{2}}$ term.

We can still pair b_0 & b_M, b_1 & b_{M-1} etc.

Therefore, the same formula as above will apply*.

For example, when $M=5$ we get

$$H(e^{j\hat{\omega}}) = e^{-j5\hat{\omega}/2} \sum_{k=0}^{2} 2j\, b_k \sin\left(\left(\tfrac{5}{2}-k\right)\hat{\omega}\right)$$

$$= 2e^{j\left(\frac{\pi}{2}-\frac{5\hat{\omega}}{2}\right)}\left(b_0 \sin\left(\tfrac{5}{2}\hat{\omega}\right) + b_1 \sin\left(\tfrac{3}{2}\hat{\omega}\right) + b_2 \sin\left(\tfrac{1}{2}\hat{\omega}\right)\right)$$

*NOTE: the upper limit on the sum is different. Since we pair $b_{\frac{M-1}{2}}$ with $b_{\frac{M+1}{2}}$, the upper limit becomes $\frac{M-1}{2}$.

McClellan, Schafer and Yoder, *Signal Processing First*, ISBN 0-13-065562-7.
Pearson Prentice Hall, Inc. Upper Saddle River, NJ 07458.

PROBLEM 7.18:

(a) $H_1(z) = H_2(z) = 1 + z^{-1} + z^{-2} + z^{-3}$

(b) $H(z) = H_1(z)H_2(z) = (1 + z^{-1} + z^{-2} + z^{-3})^2$

(c) Multiply out the product:

$$H(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 3z^{-4} + 2z^{-5} + z^{-6}$$

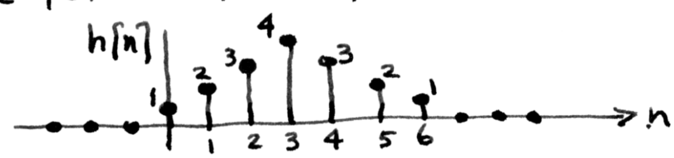
Invert term-by-term:

$$h[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2] + 4\delta[n-3] + 3\delta[n-4] + 2\delta[n-5] + \delta[n-6]$$

(d) Use the polynomial coeffs of $H(z)$ as filter coefficients:

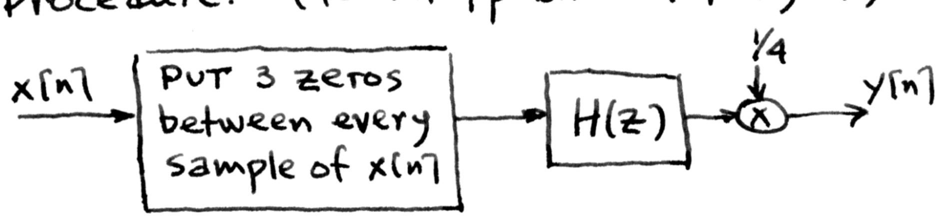
$$y[n] = x[n] + 2x[n-1] + 3x[n-2] + 4x[n-3] + 3x[n-4] + 2x[n-5] + x[n-6]$$

(e) Linear interpolation uses a triangularly shaped impulse response, which is exactly the form of $h[n]$.



There are two problems with using $h[n]$ directly: The value at the peak of the triangle is 4, not 1, so we need a scale factor of $\frac{1}{4}$. The peak value is at $n=3$, not $n=0$, so the output is time-shifted – delayed by 3.

Procedure: (to interpolate $x[n]$ by 4)



The output $y[n]$ is "$x[n]$ interpolated" but shifted by 3 samples.

McClellan, Schafer and Yoder, *Signal Processing First*, ISBN 0-13-065562-7.
Pearson Prentice Hall, Inc. Upper Saddle River, NJ 07458.

PROBLEM 7.18 (more):

(f) Prove $H_1(z) = \dfrac{1-z^{-4}}{1-z^{-1}}$

$$(1-z^{-1})H_1(z) = (1-z^{-1})(1+z^{-1}+z^{-2}+z^{-3})$$
$$= 1+z^{-1}+z^{-2}+z^{-3}-z^{-1}-z^{-2}-z^{-3}-z^{-4}$$
$$= 1-z^{-4}$$

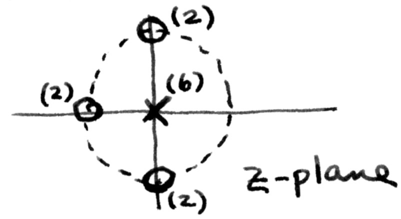
(g) $H_1(z)$ and $H_2(z)$ have the same poles & zeros so we find the poles and zeros of $H_1(z)$ and plot them "twice" in the z-plane.

$$H_1(z) = \frac{1-z^{-4}}{1-z^{-1}} = \frac{z^4-1}{z^3(z-1)} \quad \begin{array}{l} \leftarrow \text{ROOTS} = 1,-1,j,-j \\ \leftarrow \text{ROOTS} = 0,1 \end{array}$$

The numerator and denominator both have roots at $z=1$, so these cancel. We are left with 3 zeros at $z=-1,+j,-j$ and 3 poles at $z=0$.

$\Rightarrow$ $H(z)$ has 6 zeros; two each at $z=-1,+j$ and $-j$

$H(z)$ has 6 poles at $z=0$



(h) $H_1(e^{j\hat{\omega}}) = H_1(z)\big|_{z=e^{j\hat{\omega}}}$

$$= \frac{1-e^{-j4\hat{\omega}}}{1-e^{-j\hat{\omega}}} = \frac{e^{-j2\hat{\omega}}}{e^{-j\hat{\omega}/2}} \cdot \frac{(e^{j2\hat{\omega}}-e^{-j2\hat{\omega}})(\frac{1}{2j})}{(e^{j\hat{\omega}/2}-e^{-j\hat{\omega}/2})(1/2j)}$$

$$= e^{-j3\hat{\omega}/2}\,\frac{\sin(2\hat{\omega})}{\sin(\frac{1}{2}\hat{\omega})}$$

(i) $H(e^{j\hat{\omega}}) = H_1^2(e^{j\hat{\omega}}) \longrightarrow e^{-j3\hat{\omega}}\left(\dfrac{\sin(2\hat{\omega})}{\sin(\frac{1}{2}\hat{\omega})}\right)^2$