5.4 Runge-Kutta Methods

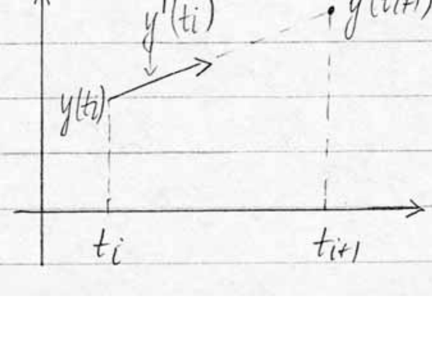
The Taylor methods have high-order local truncation error but require the computation of the derivatives of $f(t,y)$.

We introduce Runge-Kutta Methods which also have high-order local truncation error but do not require the derivatives of $f(t,y)$. For their many advantages Runge-Kutta methods are major computational tool for approximating the solutions of ODE.

1) Runge-Kutta Methods of order 2

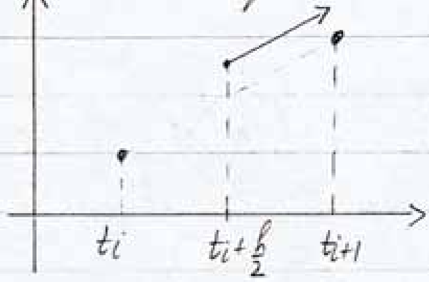
(a) Midpoint Method

In the standard Euler's method we approximate $y(t_{i+1})$ from a line through $y(t_i)$ with slope $y'(t_i)$



$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \text{Error}$$

In the Midpoint Method we improve Euler's method by using the derivative at the midpoint



$$y(t_{i+1}) = y(t_i) + h y'(t_i + \tfrac{h}{2}) + \text{Error}$$

We have from the equation:

$$y'(t_i + \tfrac{h}{2}) = f\left(t_i + \tfrac{h}{2}, y(t_i + \tfrac{h}{2})\right)$$

Since we don't know $y(t_i + \frac{h}{2})$ and we don't plan to compute it, we have to approximate with known quantities. We expand in Taylor's series

$$y(t_i + \tfrac{h}{2}) = y(t_i) + \tfrac{h}{2} y'(t_i) + \text{Error}$$

From the differential equation we have

$$y(t_i + \tfrac{h}{2}) = y(t_i) + \tfrac{h}{2} f(t_i, y_i) + \text{Error}$$

Thus, we derive the Midpoint Method

$$\begin{cases} w_0 = \alpha \\ w_{i+1} = w_i + h f\left(t_i + \frac{h}{2},\ w_i + \frac{h}{2} f(t_i, w_i)\right) \end{cases}$$

for $i = 0, 1, \ldots, N-1$

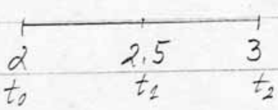
Ex #3b/280 Set up the Midpoint Method to solve the IVP

$$\begin{cases} y' = 1 + (t-y)^2 & 2 \le t \le 3 \\ y(2) = 1 \end{cases}$$

with $h = 0.5$. The exact solution is

$$y(t) = t + \frac{1}{1-t}$$

Solution: We divide the interval



$$f(t_i, w_i) = 1 + (t_i - w_i)^2$$

$$f\left(t_i + \frac{h}{2},\ w_i + \frac{h}{2} f(t_i, w_i)\right) =$$

$$1 + \left(t_i + \frac{h}{2} - \left(w_i + \frac{h}{2}\left(1 + (t_i - w_i)^2\right)\right)\right)^2$$

$$= 1 + \left(t_i + \frac{h}{2} - w_i - \frac{h}{2} - \frac{h}{2}(t_i - w_i)^2\right)^2$$

$$= 1 + \left(t_i - w_i - \frac{h}{2}(t_i - w_i)^2\right)^2$$

$$= 1 + (t_i - w_i)^2 \left(1 - \frac{h}{2}(t_i - w_i)\right)^2$$

$$= 1 + (t_i - w_i)^2 \left[1 - h(t_i - w_i) + \frac{h^2}{4}(t_i - w_i)^2\right]$$

$$= 1 + (t_i - w_i)^2 - h(t_i - w_i)^3 + \frac{h^2}{4}(t_i - w_i)^4$$

The Midpoint method is

$$\begin{cases} w_0 = 1 \\ w_{i+1} = w_i + \frac{h^3}{4}(t_i - w_i)^4 - h^2(t_i - w_i)^3 + h(t_i - w_i)^2 + h \end{cases}$$

So

$$w_1 = 1 + \frac{0.5^3}{4}(2-1)^4 - 0.5^2(2-1)^3 + 0.5(2-1)^2 + 0.5$$

$$= 1.78125$$

i	t_i	w_i	$y(t_i)$	Error
0	2	1	1	
1	2.5	1.78125	1.8333333	0.05208333
2	3	2.45506385	2.5	0.04493615

(6) Modified Euler's Method

In the Modified Euler's Method the slope of the line through $y(t_i)$ is taken to be the average of the slopes at t_i and t_{i+1}

$$\text{slope} = \frac{1}{2}\left(y'(t_i) + y'(t_{i+1})\right)$$

Thus,

$$y(t_{i+1}) = y(t_i) + \frac{1}{2}h\underbrace{\left(y'(t_i)+y'(t_{i+1})\right)}_{\text{slope is average of slopes at } t_i,\, t_{i+1}.}$$

We know that $y'(t_i) = f(t_i, y_i)$.
We also have from the ODE

$$y'(t_{i+1}) = f(t_{i+1}, y(t_{i+1}))$$

But since we still don't know $y(t_{i+1})$ (we are trying to estimate it) we have to approximate it with known quantities.

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \text{Error}$$

or

$$y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)) + \text{Error}$$

Thus,

$$y'(t_{i+1}) = f(t_{i+1}, y_i + hf(t_i, y_i))$$

Thus, the Modified Euler's Method becomes

$$\begin{cases} w_0 = \alpha \\ w_{i+1} = w_i + \frac{h}{2}\left[f(t_i, w_i) + f(t_{i+1}, w_i + hf(t_i, w_i))\right] \end{cases}$$

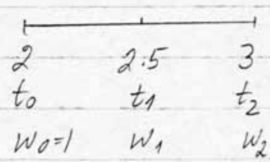
$$i = 0, 1, \ldots, N-1$$

Ex. #1b/280 Set up the Modified Euler's method to solve the IVP

$$y' = 1 + (t-y)^2 \qquad 2 \le t \le 3$$
$$y(2) = 1$$

with $h = 0.5$. The exact solution is $y(t) = t + \frac{1}{1-t}$

Solution: The interval is divided into 2 subintervals



$$f(t_i, w_i) = 1 + (t_i - w_i)^2$$

$f(t_{i+1}, w_i + hf(t_i, w_i)) =$

$f(t_{i+1}, w_i + h + h(t_i - w_i)^2) =$

$f(t_i + h, w_i + h + h(t_i - w_i)^2) =$

$1 + (t_i + h - w_i - h - h(t_i - w_i)^2)^2$

$= 1 + ((t_i - w_i) - h(t_i - w_i)^2)^2 =$

$= 1 + (t_i - w_i)^2 - 2h(t_i - w_i)^3 + h^2(t_i - w_i)^4$

Next,

$f(t_i, w_i) + f(t_{i+1}, w_i + hf(t_i, w_i)) =$

$1 + (t_i - w_i)^2 + 1 + (t_i - w_i)^2 - 2h(t_i - w_i)^3 + h^2(t_i - w_i$

$= 2 + 2(t_i - w_i)^2 - 2h(t_i - w_i)^3 + h^2(t_i - w_i)^4$

Thus, the <u>Modified Euler's Method</u> becomes

$$\begin{cases} w_0 = 1 \\ w_{i+1} = w_i + h + h(t_i - w_i)^2 - h^2(t_i - w_i)^3 + \\ \qquad + \dfrac{h^3}{2}(t_i - w_i)^4 \qquad i = 0, 1. \end{cases}$$

For $i=0$ we have

$$w_1 = w_0 + h + h(t_0 - w_0)^2 - h^2(t_0 - w_0)^3 + \frac{h^3}{2}(t_0 - w_0)^4$$

$$= 1 + 0.5 + 0.5(2-1)^2 - 0.5^2(2-1)^3 + \frac{0.5^3}{2}(2-1)^4$$

$$= 1.8125$$

We tabulate the results in the table

i	t_i	w_i	$y(t_i)$	Error
0	2	1	1	0
1	2.5	1.8125	1.8333333	0.0208333
2	3	2.4815531	2.5	0.01844685

(c) General Formula for the two-stage Runge-Kutta Methods

Set $k_1 = f(t_i, w_i)$
$k_2 = f(t_i + \alpha_2, w_i + \delta_2 k_1)$

Thus, the general 2-stage Runge-Kutta Method is

$$\begin{cases} w_0 = \alpha \\ w_{i+1} = w_i + h[a_1 k_1 + a_2 k_2] \end{cases}$$

The Midpoint Method is a special case of this formula with

$$a_1 = 0 \quad a_2 = 1 \quad \alpha_2 = \frac{h}{2} \quad \delta_2 = \frac{h}{2}$$

The Modified Euler's Method is a special case with

$$a_1 = \frac{1}{2} \quad a_2 = \frac{1}{2} \quad \alpha_2 = h \quad \delta_2 = h$$

(d) Heun's Method: Obtained from the general formula with

$$\alpha_1 = \frac{1}{4} \quad a_2 = \frac{3}{4} \quad \alpha_2 = \delta_2 = \frac{2}{3}h$$

The Heun's Method is:

$$\begin{cases} w_0 = \alpha \\ w_{i+1} = w_i + \frac{h}{4}\left[f(t_i, w_i) + 3f(t_i + \frac{2}{3}h, w_i + \frac{2}{3}hf(t_i, w_i\right. \end{cases}$$

$$i = 0, 1, \dots, N-1$$

Error in the 2-stage Runge-Kutta Method.
It can be shown that the error from 2-stage Runge-Kutta methods is $O(h^2)$ provided

$$\begin{cases} a_1 + a_2 = 1 \\ \alpha_2 = \delta_2 = \frac{h}{2a_2} \end{cases}$$

It is easy to see that this is exactly the case with the Midpoint method, Modified Euler's Method and Heun's Method

Ex #2b/280 Set up Heun's method to solve the IVP

$$\begin{cases} y' = 1 + (t-y)^2 & 2 \le t \le 3 \\ y(2) = 1 \end{cases}$$

with $h = 0.5$. The exact solution is $y(t) = t + \frac{1}{1-t}$

Solution:

$$\begin{array}{ccc} 2 & 2.5 & 3 \\ t_0 & t_1 & t_2 \end{array}$$

$$f(t_i, w_i) = 1 + (t_i - w_i)^2$$

$$f\left(t_i + \tfrac{2}{3}h,\ w_i + \tfrac{2}{3}h f(t_i, w_i)\right) =$$

$$= f\left(t_i + \tfrac{2}{3}h,\ w_i + \tfrac{2}{3}h + \tfrac{2}{3}h(t_i - w_i)^2\right)$$

$$= 1 + \left(t_i + \tfrac{2}{3}h - w_i - \tfrac{2}{3}h - \tfrac{2}{3}h(t_i - w_i)^2\right)^2$$

$$= 1 + \left(t_i - w_i - \tfrac{2}{3}h(t_i - w_i)^2\right)^2 =$$

$$= 1 + (t_i - w_i)^2 - \tfrac{4}{3}h(t_i - w_i)^3 + \tfrac{4}{9}h^2(t_i - w_i)^4$$

$$f(t_i, w_i) + 3f\left(t_i + \tfrac{2}{3}h,\ w_i + \tfrac{2}{3}h f(t_i, w_i)\right) =$$

$$= 1 + (t_i - w_i)^2 + 3 + 3(t_i - w_i)^2 - 4h(t_i - w_i)^3 + \tfrac{4}{3}h^2(t_i - w_i)^4$$

$$= 4 + 4(t_i - w_i) - 4h(t_i - w_i)^3 + \frac{4}{3}h^2(t_i - w_i)^4$$

Thus,

$$w_{i+1} = w_i + h + h(t_i - w_i) - h^2(t_i - w_i)^3 + \frac{h^3}{3}(t_i - w_i)$$

Hence, Heun's Method for this ODE is

$$\begin{cases} w_0 = 1 \\ w_{i+1} = w_i + h + h(t_i - w_i)^2 - h^2(t_i - w_i)^3 + \frac{h^3}{3}(t_i - w_i) \end{cases}$$

$$i = 0, 1$$

For $i = 0$ we have

$$w_1 = w_0 + h + h(t_0 - w_0)^2 - h^2(t_0 - w_0)^3 - \frac{h^3}{3}(t_0 - w_0)^4$$

$$= 1 + 0.5 + 0.5(2-1)^2 - 0.5^2(2-1)^3 - \frac{0.5^3}{3}(2-1)^4$$

$$= 1.79166667$$

The results are tabulated below

i	t_i	w_i	$y(t_i)$	
0	2	1	1	0
1	2.5	1.7916667	1.8333333	0.0416666333
2	3	2.4641747	2.5	0.0358253

The errors from all 3 two-stage Runge-Kutta methods are quite similar.

2) 4-stage Runge-Kutta Methods

The 4-stage Runge-Kutta Methods are most widely used for solving ODE. One of the most commonly used is the following

$$
\begin{cases}
w_0 = \alpha \\
K_1 = hf(t_i, w_i) \\
K_2 = hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2}K_1\right) \qquad i = 0, 1, \ldots, N-1 \\
K_3 = hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2}K_2\right) \\
K_4 = h\left(t_{i+1}, w_i + K_3\right) \\
w_{i+1} = w_i + \frac{1}{6}\left(K_1 + 2K_2 + 2K_3 + K_4\right)
\end{cases}
$$

Applying this method by hand is almost impossible.

Ex: Use the 4-stage Runge-Kutta method to solve the IVP

$$y' = -2t - y \qquad 0 \le t \le 0.5$$
$$y(0) = -1$$

with $h = 0.1$. Exact solution: $y(t) = 2 - 2t - 3e^{-t}$

Thus, we have

$$w_0 = -1$$

For $i = 0$, $K_1 = hf(t_0, w_0) = h(-2t_0 - w_0)$
$$= 0.1(+1) = 0.1$$

$$K_2 = 0.1\left[-2\left(t_0 + \frac{h}{2}\right) - w_0 - \frac{1}{2}K_1\right] =$$
$$= 0.1\left[-2 \cdot \frac{0.1}{2} + 1 - 0.05\right] =$$
$$= 0.1\left[1 - 0.15\right] = (0.1)(0.85) = 0.085$$

$$K_3 = 0.1 f\left(t_0 + \frac{h}{2}, w_0 + \frac{1}{2}K_2\right) =$$
$$= 0.1\left[-2\left(t_0 + \frac{h}{2}\right) - w_0 - \frac{1}{2}K_2\right] =$$
$$= 0.1\left[-0.1 + 1 - 0.0425\right]$$
$$= (0.1)(0.8575) = 0.08575$$

$$K_4 = 0.1 f(t_1, w_0 + K_3) =$$
$$= 0.1\left[-2t_1 - w_0 - K_3\right] = 0.1\left[-0.2 + 1 - 0.08575\right]$$
$$= 0.1(0.71425) = 0.071425$$

$$w_1 = -1 + \frac{1}{6}(0.1 + 2 \cdot 0.085 + 2 \cdot 0.08575 + 0.071425)$$

$$= -0.9145125$$

Thus, we obtain the following result

t_i	k_1	k_2	k_3	k_4	w_i	$y(t_i)$	Error
0	0.1	0.085	0.0858	0.0714	-1	-1	0
0.1	0.0715	0.0579	0.0586	0.0456	-0.91451	-0.91451	0
0.2	0.0456	0.0333	0.034	0.0222	-0.85619	-0.85619	0
0.3	0.0222	0.0111	0.0117	0.0011	-0.82246	-0.82245	0.00001
0.4	0.0011	-0.009	-0.0085	-0.0181	-0.81096	-0.81096	0
0.5	-0.018	-0.0271	-0.0267	-0.0354	-0.81959	-0.81959	0

With 5 digits after the decimal point we don't see any error!

This 4-stage Runge-Kutta method has a local truncation error $O(h^4)$, provided $y(t)$ has 5 continuous derivatives

3) Computational effort of Runge-Kutta Methods.

The computational effort necessary to use Runge-Kutta methods is measured by the number of evaluations of $f(t,y)$.

Thus, Euler's method

$$w_{i+1} = w_i + hf(t_i, w_i)$$

has one evaluation per step. Notice that its local truncation error is $O(h)$. The 2-stage Runge-Kutta methods have 2 evaluations per step and local truncation error $O(h^2)$. The 4-stage Runge-Kutta method has 4 evaluations and local truncation error $O(h^4)$.

This means that while we make 1 step with the 4-stage method (4 evaluations) we can make 2 steps with a 2-stage method (with step $\frac{h}{2}$) and 4 steps with Euler's method (with step $\frac{h}{4}$). Comparing the methods when they require the sam. computational effort we see that the 4-stage Runge-Kutta method is clearly superior (see table 5.8 in the book).