## PROBLEM 8.1:

$$y[n] = \sqrt{2}\, y[n-1] - y[n-2] + x[n] \qquad x[n] = \delta[n]$$

"At rest" condition $\Rightarrow$ $y[n] = 0$ for $n < 0$.

$$y[0] = \sqrt{2}\, y[-1] - y[-2] + x[0] = (\sqrt{2})0 - 0 + 1 = 1$$

$$y[1] = \sqrt{2}\, y[0] - y[-1] + x[1] = (\sqrt{2})1 - 0 + 0 = \sqrt{2}$$

$$y[2] = \sqrt{2}\, y[1] - y[0] + x[2] = (\sqrt{2})\sqrt{2} - 1 + 0 = 1$$

$$y[3] = (\sqrt{2})1 - \sqrt{2} + 0 = 0$$

$$y[4] = (\sqrt{2})0 - 1 + 0 = -1$$

The general formula is

$$y[n] = A_1 (r_1)^n + A_2 (r_2)^n \qquad \text{for } n \geq 0$$

where $r_1$ & $r_2$ are the poles.

$$H(z) = \frac{1}{1 - \sqrt{2}\, z^{-1} + z^{-2}}$$

Poles are roots of denominator:

$$\frac{\sqrt{2} \pm \sqrt{2-4}}{2} = \frac{\sqrt{2}}{2} \pm j\frac{\sqrt{2}}{2} = e^{\pm j\pi/4}$$

$$y[n] = A_1 (e^{j\pi/4})^n + A_2 e^{-j\frac{\pi}{4}n}$$

Now, we evaluate $A_1$ & $A_2$ from known values of $y[n]$. We use $n=2$ and $n=4$

$$y[2] = 1 = A_1 e^{j\pi/2} + A_2 e^{-j\pi/2} = jA_1 - jA_2$$

$$y[4] = -1 = A_1 e^{j\pi} + A_2 e^{-j\pi} = -A_1 - A_2$$

Solve the simultaneous equations:

$$1 - j = -2j A_2 \quad \text{and} \quad 1 + j = 2j A_1 \Rightarrow A_1 = \frac{1+j}{2j} = \frac{1}{2} - j\frac{1}{2}$$

$$A_2 = A_1^* \qquad A_1 = \frac{\sqrt{2}}{2} e^{-j\pi/4}$$

$$y[n] = \frac{\sqrt{2}}{2} e^{-j\pi/4} e^{j\frac{\pi}{4}n} + \frac{\sqrt{2}}{2} e^{+j\pi/4} e^{-j\pi n/4} \qquad \underline{\text{for } n \geq 0}$$

$$= \sqrt{2}\cos\left(\frac{\pi}{4}n - \frac{\pi}{4}\right) = \sqrt{2}\cos\left(\frac{\pi}{4}(n-1)\right)$$

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## PROBLEM 8.2:

First of all, tabulate some values of $h[n]$.

$$\underset{h[n]}{y[n]} = y[n-1] + y[n-2] + \underset{\delta[n]}{x[n]}$$

When $x[n] = \delta[n]$ we use $h[n]$ in place of $y[n]$.

| $n$ | $<0$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|---|
| $x[n] = \delta[n]$ | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $h[n]$ | 0 | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 |

NOTE: ADD THE LAST 2 OUTPUTS

Now we use z-Transform to get the formula for $n \geq 0$. For $n<0$, $h[n]=0$.

$$Y(z) = z^{-1}Y(z) + z^{-2}Y(z) + X(z)$$

$$(1 - z^{-1} - z^{-2})Y(z) = X(z)$$

$$H(z) = Y(z)/X(z) = \frac{1}{1 - z^{-1} - z^{-2}}$$

The poles are at: $z = \dfrac{1 \pm \sqrt{1-4(1)(-1)}}{2} = \dfrac{1 \pm \sqrt{5}}{2}$

To avoid the inverse z-Transform (for now), we use the idea that the output is "pole-to-the-n"

$$\Rightarrow \quad h[n] = K_1\left(\frac{1+\sqrt{5}}{2}\right)^n u[n] + K_2\left(\frac{1-\sqrt{5}}{2}\right)^n u[n]$$

VALID FOR $n \geq 0$

Use $h[0]$ & $h[1]$:

$$1 = h[0] = K_1 + K_2$$

$$1 = h[1] = \left(\frac{1+\sqrt{5}}{2}\right)K_1 + \left(\frac{1-\sqrt{5}}{2}\right)K_2$$

Solve for $K_1$ & $K_2$

$$K_1 = \frac{1+\sqrt{5}}{2\sqrt{5}}$$

$$K_2 = -\frac{1-\sqrt{5}}{2\sqrt{5}}$$

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## PROBLEM 8.3:

$$h[n] = 5(0.8)^n u[n] \quad \& \quad x[n] = \delta[n] - \alpha\delta[n-5]$$

$$y[n] = h[n] * x[n] = 5(0.8)^n u[n] - 5\alpha(0.8)^{n-5} u[n-5]$$

Want $y[n] = 0$ for $n \geq 5$

$$\Rightarrow \quad 5(0.8)^n - 5\alpha(0.8)^{n-5} = 0$$

$$\Rightarrow \quad 5 = 5\alpha(0.8)^{-5}$$

$$\Rightarrow \quad \alpha = (0.8)^5 \cong 0.3277$$

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## PROBLEM 8.4:

$$y[n] = \frac{1}{2}y[n-1] + \frac{1}{3}y[n-2] - x[n] + 3x[n-1] - 2x[n-2]$$

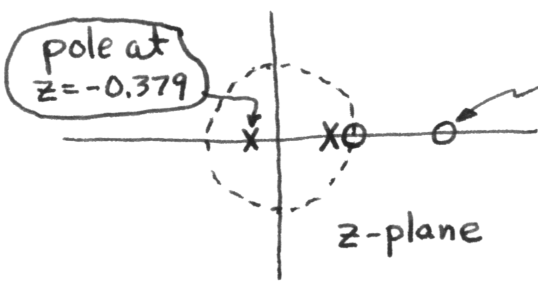
$$Y(z) = \frac{1}{2}z^{-1}Y(z) + \frac{1}{3}z^{-2}Y(z) - X(z) + 3z^{-1}X(z) - 2z^{-2}X(z)$$

$$\left(1 - \frac{1}{2}z^{-1} - \frac{1}{3}z^{-2}\right)Y(z) = \left(-1 + 3z^{-1} - 2z^{-2}\right)X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{-1 + 3z^{-1} - 2z^{-2}}{1 - \frac{1}{2}z^{-1} - \frac{1}{3}z^{-2}}$$

Change to positive powers of $z$ to find roots.

$$H(z) = -\frac{z^2 - 3z + 2}{z^2 - \frac{1}{2}z - \frac{1}{3}} = -\frac{(z-2)(z-1)}{(z-0.879)(z+0.379)}$$




For the second system only the signs on $y[n-2]$ and $x[n-2]$ change, so we can write $H(z)$ immediately:

$$H(z) = -\frac{1 - 3z^{-1} - 2z^{-2}}{1 - \frac{1}{2}z^{-1} + \frac{1}{3}z^{-2}} = -\frac{z^2 - 3z - 2}{z^2 - \frac{1}{2}z + \frac{1}{3}}$$

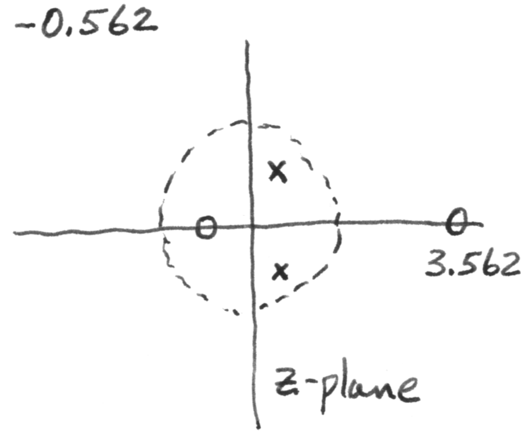
ZEROS: $\frac{3 \pm \sqrt{9+8}}{2} = 3.562, -0.562$

POLES: $0.25 \pm j0.52$

$= 0.5774e^{\pm j0.357\pi}$

ANGLE is $\pm 64.34°$

or $\pm 1.123$ rads.



## PROBLEM 8.5:

$$y[n] = \tfrac{1}{2}y[n-1] - \tfrac{1}{3}y[n-2] - x[n]$$

$$Y(z) = \tfrac{1}{2}z^{-1}Y(z) - \tfrac{1}{3}z^{-2}Y(z) - X(z)$$

$$\left(1-\tfrac{1}{2}z^{-1}+\tfrac{1}{3}z^{-2}\right)Y(z) = -X(z)$$

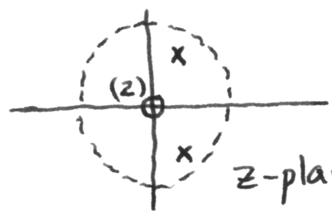
$$H(z) = \frac{Y(z)}{X(z)} = \frac{-1}{1-\frac{1}{2}z^{-1}+\frac{1}{3}z^{-2}}$$

Change to positive powers of z when finding poles and zeros.

$$H(z) = \frac{-z^2}{z^2-\frac{1}{2}z+\frac{1}{3}}$$

poles are at
$z = 0.25 \pm j0.52$
$= 0.5774e^{\pm j0.357\pi}$
ANGLE $= \pm 64.34°$
or $\pm 1.123$ rads

Numerator is $z^2$, so we have two zeros at $z=0$.




z-plane

---

$$y[n] = \tfrac{1}{2}y[n-1] - \tfrac{1}{3}y[n-2] - x[n-2]$$

$$H(z) = \frac{-z^{-2}}{1-\frac{1}{2}z^{-1}+\frac{1}{3}z^{-2}} = \frac{-1}{z^2-\frac{1}{2}z+\frac{1}{3}}$$ ← Same poles

If we take $\lim_{z\to\infty} H(z)$ we get $H(z)\to\frac{1}{z^2}$ so we have 2 zeros at $z=\infty$

---

$$y[n] = \tfrac{1}{2}y[n-1] - \tfrac{1}{3}y[n-2] - x[n-4]$$

$$H(z) = \frac{-z^{-4}}{1-\frac{1}{2}z^{-1}+\frac{1}{3}z^{-2}} = \frac{-1}{z^2\left(z^2-\frac{1}{2}z+\frac{1}{3}\right)}$$

Now $H(z)\to 1/z^4$ as $z\to\infty$, so we have 4 zeros at $z=\infty$

We have 4 poles. The same two as above, plus 2 more poles at $z=0$.

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## PROBLEM 8.6:

(a) $y[n] = -\frac{1}{2} y[n-1] + x[n]$

If re-arranged:

$$y[n] + \frac{1}{2} y[n-1] = x[n]$$

$\underline{a} = [1 \;\; +\tfrac{1}{2}];$
$b = 1$

$$H(z) = \frac{1}{1+\frac{1}{2}z^{-1}} = \frac{z}{z+1/2}$$

ZERO @ $z=0$
POLE @ $z=-\frac{1}{2}$

(b) Do this by making a table.

| $n$ | $x[n]$ | $y[n]$ | |
|---|---|---|---|
| $<0$ | 0 | 0 | |
| 0 | 1 | 1 | $y[0] = -\frac{1}{2} y[-1] + x[0] = 1$ |
| 1 | 1 | $1/2$ | $y[1] = -\frac{1}{2}(1) + 1 = 1/2$ |
| 2 | 1 | $3/4$ | $y[2] = -\frac{1}{2}(1/2) + 1 = 3/4$ |
| 3 | 0 | $-3/8$ | $y[3] = -1/2\,(3/4) + 0$ |
| 4 | 0 | $3/16$ | $y[4] = -1/2\,(-3/8) + 0 -$ |
| 5 | 0 | $-3/32$ | |
| 6 | 0 | $3/64$ | |
| 7 | 0 | $-3/128$ | |
| 8 | 0 | $3/256$ | |

$$\therefore \quad y[n] = \begin{cases} 0 & n<0 \\ 1 & n=0 \\ 1/2 & n=1 \\ 3/(-2)^n & n \geq 2 \end{cases}$$

RESPONSE BEHAVES LIKE

$3\left(\frac{-1}{2}\right)^n$ for $n \geq 2$

POLE

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## PROBLEM 8.7:

$$y[n] = -0.8y[n-1] + 0.8x[n] + x[n-1]$$

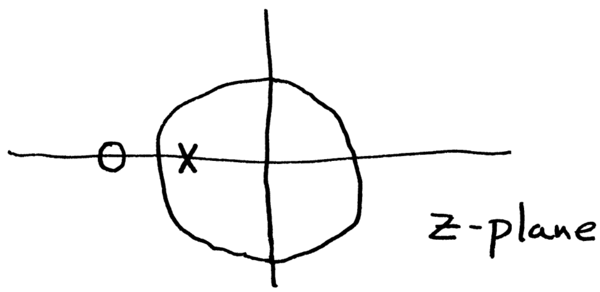
(a) $$Y(z) = -0.8z^{-1}Y(z) + 0.8X(z) + z^{-1}X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{0.8 + z^{-1}}{1 + 0.8z^{-1}}$$

$$= \frac{0.8z + 1}{z + 0.8}$$

(b) Pole at: $z + 0.8 = 0 \Rightarrow z = -0.8$

Zero at: $0.8z + 1 = 0 \Rightarrow z = -1/0.8 = -1.25$




(c) $$H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}}$$

$$= \frac{0.8 + e^{-j\hat{\omega}}}{1 + 0.8e^{-j\hat{\omega}}}$$

(d) $$|H(e^{j\hat{\omega}})|^2 = H(e^{j\hat{\omega}})H^*(e^{j\hat{\omega}})$$

$$= \frac{0.8 + e^{-j\hat{\omega}}}{1 + 0.8e^{-j\hat{\omega}}} \cdot \frac{0.8 + e^{j\hat{\omega}}}{1 + 0.8e^{j\hat{\omega}}}$$

$$= \frac{0.64 + 0.8e^{-j\hat{\omega}} + 0.8e^{j\hat{\omega}} + 1}{1 + 0.8e^{-j\hat{\omega}} + 0.8e^{j\hat{\omega}} + 0.64}$$

$$= \frac{1.64 + 1.6\cos\hat{\omega}}{1.64 + 1.6\cos\hat{\omega}}$$

$$= 1$$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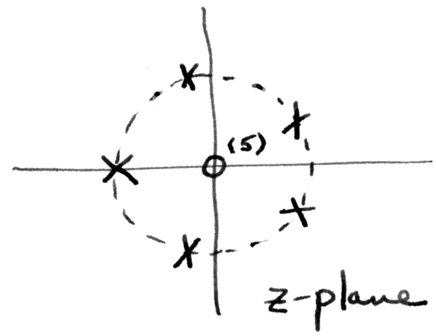
## PROBLEM 8.8:

(a) $H(z) = \dfrac{1}{1+z^{-5}}$

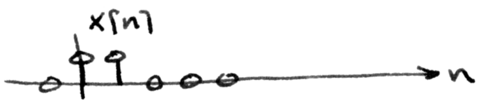
(b) FIVE POLES. Find roots of

$z^5 + 1 = 0$

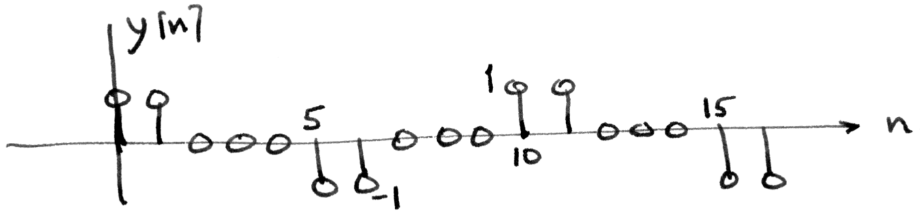
$z = \underset{36^\circ}{e^{j\pi/5}},\ \underset{108^\circ}{e^{j3\pi/5}},\ e^{j\pi},\ e^{-j\pi/5},\ e^{-j3\pi/5}$




(c)




| n | <0 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| y[n] | 0 | 1 | 1 | 0 | 0 | 0 | -1 | -1 | 0 | 0 | 0 | 1 | 1 | 0 |




(d) PERIOD = 10

which can be determined from the plot above.

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## PROBLEM 8.9:

$$y[n] = -0.9\,y[n-6] + x[n]$$

(a) $$Y(z) = -0.9 z^{-6} Y(z) + X(z)$$

$$H(z) = \frac{1}{1+0.9z^{-6}} = \frac{z^6}{z^6+0.9}$$

SIX ZEROS AT $z=0$

(b) Poles are found as the solutions to

$$z^6 + 0.9 = 0$$

This involves the "roots of unity"

$$z^6 = -0.9 = 0.9e^{j\pi}e^{j2\pi\ell} \qquad \ell = 0,1,2,3,4,5$$

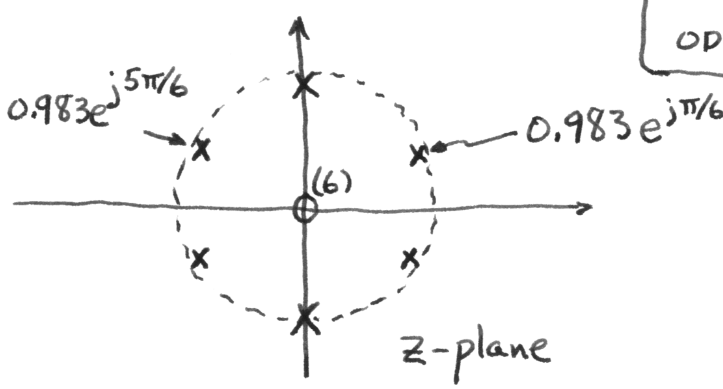
$$z = \sqrt[6]{0.9}\, e^{j\pi/6} e^{j\pi\ell/3}$$

$$= 0.983\, e^{j\pi(2\ell+1)/6}$$

ANGLES ARE:

$\frac{\pi}{6}, \frac{3\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{9\pi}{6}, \frac{11\pi}{6}$

ODD MULTIPLES of 30°

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# PROBLEM 8.10:

$$y[n] = -\tfrac{1}{2}y[n-1] + x[n]$$

(a) $Y(z) = -\frac{1}{2}z^{-1}Y(z) + X(z)$

$$(1+\tfrac{1}{2}z^{-1})Y(z) = X(z) \implies H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1+\frac{1}{2}z^{-1}}$$

To find poles & zeros change to positive powers of $z$.

$$H(z) = \frac{z}{z+1/2} \implies \begin{array}{l}\text{1 zero at } z=0 \\ \text{one pole at } z=-1/2.\end{array}$$

(b) The impulse response of the system is the inverse transform of $H(z)$:

$$H(z) = \frac{1}{1+\frac{1}{2}z^{-1}} \longrightarrow h[n] = (-\tfrac{1}{2})^n u[n]$$

To get the output when $x[n] = \delta[n] + \delta[n-1] + \delta[n-2]$ use superposition.

$$\begin{aligned} y[n] &= h[n] + h[n-1] + h[n-2] \\ &= (-\tfrac{1}{2})^n u[n] + (-\tfrac{1}{2})^{n-1}u[n-1] + (-\tfrac{1}{2})^{n-2}u[n-2] \end{aligned}$$

For $n=0$, $y[0] = 1+0+0 = 1$

For $n=1$, $y[1] = -\frac{1}{2}+1+0 = 1/2$

For $n\geq 2$, $y[n] = (-\frac{1}{2})^n + (-\frac{1}{2})^{n-1} + (-\frac{1}{2})^{n-2}$

$$= (-\tfrac{1}{2})^n(1-2+4) = 3(-\tfrac{1}{2})^n$$

Formula for $y[n]$:

$$y[n] = \delta[n] + \tfrac{1}{2}\delta[n-1] + 3(-\tfrac{1}{2})^n u[n-2]$$

## PROBLEM 8.11:

(a) Use long division as in Example 8.11

$$\begin{array}{r} -1.3 \\ 0.77z^{-1}+1 \overline{) -z^{-1}+1} \\ -z^{-1}-1.3 \\ \hline 2.3 \end{array}$$

QUOTIENT: $-1.3$

REMAINDER: $2.3$

NOTE: $\frac{1}{0.77} = 1.2987 \approx 1.3$

$$H_a(z) = -1.3 + \frac{2.3}{1+0.77z^{-1}}$$

Use z-Transform pair: $\frac{b}{1-az^{-1}} \longleftrightarrow b\,a^n u[n]$

$$h_a[n] = -1.3\,\delta[n] + 2.3(-0.77)^n u[n]$$

---

(b) Use long division:

$$H_b(z) = \frac{1+0.8z^{-1}}{1-0.9z^{-1}} = -\frac{8}{9} + \frac{17/9}{1-0.9z^{-1}}$$

$$\Rightarrow h_b[n] = -\frac{8}{9}\delta[n] + \frac{17}{9}(0.9)^n u[n]$$

---

(c) Use the shifting property: $z^{-n_0}H(z) \longleftrightarrow h[n-n_0]$

$$H_c(z) = z^{-2}\left(\frac{1}{1-0.9z^{-2}}\right) = z^{-2}G_c(z)$$

$g_c[n] = (0.9)^n u[n]$.

$$h_c[n] = g_c[n-2] = (0.9)^{n-2}u[n-2]$$

This signal starts at $n=2$

---

(d) This is an FIR filter.

Invert term by term:

$$H_d(z) = 1 - z^{-1} + 2z^{-3} - 3z^{-4}$$

$\delta[n]$ $\quad -\delta[n-1]$ $\quad 2\delta[n-3]$ $\quad -3\delta[n-4]$

$$h_d[n] = \delta[n] - \delta[n-1] + 2\delta[n-3] - 3\delta[n-4]$$

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## PROBLEM 8.12:

(a) $$X_a(z) = \frac{1 - z^{-1}}{1 - \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}} = \frac{K_1}{1 - \frac{1}{2}z^{-1}} + \frac{K_2}{1 + \frac{1}{3}z^{-1}}$$

$$K_1 = \left.\frac{1 - z^{-1}}{1 + \frac{1}{3}z^{-1}}\right|_{z=\frac{1}{2}} = \frac{1 - 2}{1 + 2/3} = -3/5$$

$$K_2 = \left.\frac{1 - z^{-1}}{1 - \frac{1}{2}z^{-1}}\right|_{z=-\frac{1}{3}} = \frac{1 - (-3)}{1 + 3/2} = 8/5$$

$$x_a[n] = -\frac{3}{5}\left(\frac{1}{2}\right)^n u[n] + \frac{8}{5}\left(-\frac{1}{3}\right)^n u[n]$$

(b) $$X_b(z) = \frac{1 + z^{-2}}{1 + 0.9z^{-1} + 0.81z^{-2}}$$

$$= K_0 + \frac{K_1}{1 - 0.9e^{j2\pi/3}z^{-1}} + \frac{K_2}{1 - 0.9e^{-j2\pi/3}z^{-1}}$$

$$K_0 = 1/0.81 = 1.2346$$

$$K_1 = K_2^* = 0.6556\, e^{j0.557\pi} \leftarrow \boxed{1.75 \text{ rads}}$$

$$x_b[n] = 1.2346\,\delta[n] + 0.6556\, e^{j0.557\pi}(0.9)^n e^{j2\pi n/3} u[n] + 0.6556\, e^{-j0.557\pi}(0.9)^n e^{-j2\pi n/3} u[n]$$

$$= 1.2346\,\delta[n] + 1.311\,(0.9)^n \cos\left(\frac{2\pi}{3}n + 0.557\pi\right) u[n].$$

(c) $$X_c(z) = \frac{1 + z^{-1}}{1 - 0.1z^{-1} - 0.72z^{-2}} = \frac{K_1}{1 - 0.9z^{-1}} + \frac{K_2}{1 + 0.8z^{-1}}$$

$$K_1 = \left.\frac{1 + z^{-1}}{1 + 0.8z^{-1}}\right|_{z=0.9} = \frac{1 + 10/9}{1 + \frac{8}{10}\cdot\frac{10}{9}} = \frac{19}{17} = 1.1176$$

$$K_2 = \left.\frac{1 + z^{-1}}{1 - 0.9z^{-1}}\right|_{z=-0.8} = \frac{1 - 10/8}{1 + \frac{9}{10}\cdot\frac{10}{8}} = \frac{-2}{17} = -0.1176$$

$$x_c[n] = \frac{19}{17}\left(\frac{9}{10}\right)^n u[n] - \frac{2}{17}\left(-\frac{4}{5}\right)^n u[n]$$

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## PROBLEM 8.13:

Characterize each system ($S_1 \to S_7$)

$S_1$: $H_1(z) = \dfrac{\frac{1}{2} + \frac{1}{2}z^{-1}}{1 - 0.9z^{-1}}$ $\Rightarrow$ pole at $z = 0.9$
zero at $z = -1$

$H_1(e^{j\hat{\omega}})$ is a LPF with a null at $\hat{\omega} = \pi$.

$S_2$: $H_2(z) = \dfrac{9 + 10z^{-1}}{1 + 0.9z^{-1}}$ $\Rightarrow$ pole at $z = -0.9$
zero at $z = -10/9$

$H_2(e^{j\hat{\omega}})$ is an all-pass filter

$S_3$: $H_3(z) = \dfrac{\frac{1}{2}(1 - z^{-1})}{1 + 0.9z^{-1}}$ $\Rightarrow$ pole at $z = -0.9$
zero at $z = 1$

$H_3(e^{j\hat{\omega}})$ is a HPF with a null at $\hat{\omega} = 0$.

$S_4$: $H_4(z) = \frac{1}{4}(1 + 4z^{-1} + 6z^{-2} + 4z^{-3} + z^{-4})$

$= \frac{1}{4}(1 + z^{-1})^4$ $\Rightarrow$ 4 zeros at $z = -1$

$H_4(e^{j\hat{\omega}})$ is a LPF with null at $\hat{\omega} = \pi$.

DC value: $H_4(e^{j0}) = 4$.

$S_5$: $H_5(z) = 1 - z^{-1} + z^{-2} - z^{-3} + z^{-4} = \dfrac{1 + z^{-5}}{1 + z^{-1}}$

has 4 zeros around the unit circle.
No zero at $z = -1$; others at $e^{j(2\pi k/5 - \pi/5)}$

$H_5(e^{j\hat{\omega}})$ is a HPF with nulls at $\hat{\omega} = \pm\frac{\pi}{5}, \pm\frac{3\pi}{5}$

$S_6$: $H_6(z) = 1 + z^{-1} + z^{-2} + z^{-3} = \dfrac{1 - z^{-4}}{1 - z^{-1}}$

has 3 zeros around the unit circle at $z = \pm j, -1$

$H_6(e^{j\hat{\omega}})$ is a LPF with nulls at $\hat{\omega} = \pm\frac{\pi}{2}, \pi$

$S_7$: $H_7(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5} = \dfrac{1 - z^{-6}}{1 - z^{-1}}$

has 5 zeros around the unit circle at $z = e^{j\pi k/3}$

$H_7(e^{j\hat{\omega}})$ is a LPF with nulls at $\hat{\omega} = \pm\frac{\pi}{3}, \pm 2\frac{\pi}{3}, \pi$

PZ#1: $S_7$ PZ#3: $S_2$ PZ#5: $S_5$

PZ#2: $S_1$ PZ#4: $S_6$ PZ#6: $S_3$

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## PROBLEM 8.14:

Characterize each system ($S_1 \to S_7$)

$S_1$: $H_1(z) = \dfrac{\frac{1}{2} + \frac{1}{2}z^{-1}}{1 - 0.9z^{-1}}$ $\Rightarrow$ pole at $z = 0.9$
zero at $z = -1$

$H_1(e^{j\hat{\omega}})$ is a LPF with a null at $\hat{\omega} = \pi$.

$S_2$: $H_2(z) = \dfrac{9 + 10z^{-1}}{1 + 0.9z^{-1}}$ $\Rightarrow$ pole at $z = -0.9$
zero at $z = -10/9$

$H_2(e^{j\hat{\omega}})$ is an all-pass filter

$S_3$: $H_3(z) = \dfrac{\frac{1}{2}(1 - z^{-1})}{1 + 0.9z^{-1}}$ $\Rightarrow$ pole at $z = -0.9$
zero at $z = 1$

$H_3(e^{j\hat{\omega}})$ is a HPF with a null at $\hat{\omega} = 0$.

$S_4$: $H_4(z) = \frac{1}{4}(1 + 4z^{-1} + 6z^{-2} + 4z^{-3} + z^{-4})$

$= \frac{1}{4}(1 + z^{-1})^4$ $\Rightarrow$ 4 zeros at $z = -1$

$H_4(e^{j\hat{\omega}})$ is a LPF with null at $\hat{\omega} = \pi$.

DC value: $H_4(e^{j0}) = 4$.

$S_5$: $H_5(z) = 1 - z^{-1} + z^{-2} - z^{-3} + z^{-4} = \dfrac{1 + z^{-5}}{1 + z^{-1}}$

has 4 zeros around the unit circle.
No zero at $z = -1$; others at $e^{j(2\pi k/5 - \pi/5)}$

$H_5(e^{j\hat{\omega}})$ is a HPF with nulls at $\hat{\omega} = \pm\frac{\pi}{5}, \pm\frac{3\pi}{5}$

$S_6$: $H_6(z) = 1 + z^{-1} + z^{-2} + z^{-3} = \dfrac{1 - z^{-4}}{1 - z^{-1}}$

has 3 zeros around the unit circle at $z = \pm j, -1$

$H_6(e^{j\hat{\omega}})$ is a LPF with nulls at $\hat{\omega} = \pm\frac{\pi}{2}, \pi$

$S_7$: $H_7(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5} = \dfrac{1 - z^{-6}}{1 - z^{-1}}$

has 5 zeros around the unit circle at $z = e^{j\pi k/3}$

$H_7(e^{j\hat{\omega}})$ is a LPF with nulls at $\hat{\omega} = \pm\frac{\pi}{3}, \pm 2\frac{\pi}{3}, \pi$

(A) $S_1$ (C) $S_6$ (E) $S_5$

(B) $S_3$ (D) $S_2$ (F) $S_4$

## PROBLEM 8.15:

$$y[n] = \tfrac{1}{2} y[n-1] + x[n] \implies H(z) = \frac{1}{1 - \frac{1}{2} z^{-1}}$$

(a) Input is $u[n] \implies X(z) = \dfrac{1}{1 - z^{-1}}$

$$Y(z) = H(z)X(z) = \frac{1}{(1 - z^{-1})(1 - \frac{1}{2} z^{-1})}$$

Do a partial fraction expansion:

$$Y(z) = \frac{K_1}{1 - z^{-1}} + \frac{K_2}{1 - \frac{1}{2} z^{-1}}$$

$$K_1 = \left.\frac{1}{1 - \frac{1}{2} z^{-1}}\right|_{z=1} = 2$$

$$K_2 = \left.\frac{1}{1 - z^{-1}}\right|_{z=\frac{1}{2}} = \frac{1}{1-2} = -1$$

$$= \frac{2}{1 - z^{-1}} - \frac{1}{1 - \frac{1}{2} z^{-1}}$$

$$y[n] = 2u[n] - \left(\tfrac{1}{2}\right)^n u[n]$$

---

(b) $x[n] = e^{j(\pi/4)n} u[n] \implies X(z) = \dfrac{1}{1 - e^{j\pi/4} z^{-1}}$

$$Y(z) = H(z)X(z) = \frac{1}{(1 - \frac{1}{2} z^{-1})(1 - e^{j\pi/4} z^{-1})}$$

Partial Fraction Expansion:

$$Y(z) = \frac{A_1}{1 - \frac{1}{2} z^{-1}} + \frac{A_2}{1 - e^{j\pi/4} z^{-1}}$$

$$A_1 = \left.\frac{1}{1 - e^{j\pi/4} z^{-1}}\right|_{z=1/2} \cong 0.68 e^{j.59\pi}$$

$$A_2 = \left.\frac{1}{1 - \frac{1}{2} z^{-1}}\right|_{z=e^{j\pi/4}} \cong 1.36 e^{-j.16\pi}$$

$$y[n] = \underbrace{0.68 e^{j0.59\pi} \left(\tfrac{1}{2}\right)^n u[n]}_{\text{This term dies out}} + \underbrace{1.36 e^{-j0.16\pi} (e^{j\pi n/4}) u[n]}_{\text{Steady-state term}}.$$

(c) $\mathcal{H}(\hat{\omega})\big|_{\hat{\omega}=\pi/4} = H(e^{j\pi/4}) = H(z)\big|_{z=e^{j\pi/4}}$

$$H(e^{j\pi/4}) = \frac{1}{1 - \frac{1}{2} e^{-j\pi/4}} \cong 1.36 e^{-j0.16\pi}$$

which is the same as $A_2$

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## PROBLEM 8.16:

PZ#1: zero at $z=1 \Rightarrow$ zero at $\hat{\omega}=0$
only (D) has a zero at DC

PZ#2: pole on real axis but far from $z=1$.
$\Rightarrow$ LPF with very wide passband. (B)

PZ#3: pole very close to $z=1 \Rightarrow$ narrow LPF
also, zero at $z=-1 \Rightarrow$ zero at $\hat{\omega}=\pi$ (A)

PZ#4: pole angles are approximately $\pm\pi/6$
$\Rightarrow$ peaks near $\hat{\omega}=\pm\pi/6$ (E)

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## PROBLEM 8.17:

PZ#1: pole angles are approximately $\pm 2\pi/6$
$\Rightarrow$ oscillation period of 6, with decay (N)

PZ#2: pole at approximately $z = 0.95$ (M)
$\Rightarrow (0.95)^n$ slow decay

PZ#3: pole at approximately $z = -0.95$ (J)
$\Rightarrow (-0.95)^n \Rightarrow$ changing sign

PZ#4: pole at approximately $z = 0.4$ (L)
$\Rightarrow (0.4)^n$ rapid decay

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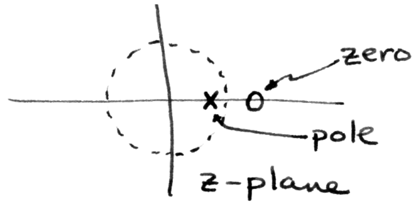
## PROBLEM 8.18:

(a) $H(z) = \dfrac{-0.8 + z^{-1}}{1 - 0.8z^{-1}}$ ← BY PICKING THE COEFFS FROM THE DIFF. EQN.

(b) POLE @ $z = 0.8$

ZERO @ $z = 1/0.8$
$= 1.25$




(c) $H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}} = \dfrac{-0.8 + e^{-j\hat{\omega}}}{1 - 0.8e^{-j\hat{\omega}}}$

(d) $|H(e^{j\hat{\omega}})|^2 = H(e^{j\hat{\omega}})H^*(e^{j\hat{\omega}})$ ← MULTIPLY BY CONJUGATE

$$= \frac{(-0.8 + e^{-j\hat{\omega}})(-0.8 + e^{+j\hat{\omega}})}{(1 - 0.8e^{-j\hat{\omega}})(1 - 0.8e^{+j\hat{\omega}})}$$

$$= \frac{.64 + 1 - 0.8e^{-j\hat{\omega}} - 0.8e^{+j\hat{\omega}}}{1 + .64 - 0.8e^{-j\hat{\omega}} - 0.8e^{+j\hat{\omega}}}$$

$$= \frac{1.64 - 1.6\cos\hat{\omega}}{1.64 - 1.6\cos\hat{\omega}} \qquad \therefore \quad |H(e^{j\hat{\omega}})|^2 = 1$$

(e) $x[n] = 4 + \cos(\frac{\pi}{4}n) - 3\cos(\frac{2\pi}{3}n)$

Need $H(e^{j0})$    Need $H(e^{j\pi/4})$    Need $H(e^{j2\pi/3})$

Since $|H(e^{j\hat{\omega}})| = 1$ for all freqs, only the phase of the cosine terms will change. Also, the phase at $\hat{\omega} = 0$ is zero, so

$$y[n] = 4 + \cos\left(\tfrac{\pi}{4}n + \angle H(e^{j\pi/4})\right) - 3\cos\left(\tfrac{2\pi}{3}n + \angle H(e^{j2\pi/3})\right)$$

$\angle H(e^{j\pi/4}) = -149.97^\circ = -2.617 \text{ rads} = -0.833\pi \text{ rads}$

$\angle H(e^{j2\pi/3}) = -172.66^\circ = -3.013 \text{ rads} = -0.959\pi \text{ rads}$

## PROBLEM 8.19:

Multiply out $H(z)$

$$H(z) = \frac{(1-z^{-1})(1-jz^{-1})(1+jz^{-1})}{(1-0.9e^{j2\pi/3}z^{-1})(1-0.9e^{-j2\pi/3}z^{-1})}$$

$$= \frac{(1-z^{-1})(1+z^{-2})}{1-2(0.9)\cos(2\pi/3)z^{-1}+(0.9)^2 z^{-2}}$$

$$= \frac{1-z^{-1}+z^{-2}-z^{-3}}{1-0.9z^{-1}+0.81z^{-2}}$$

(a) Use the numerator & denominator polynomial coefficients as filter coefficients:

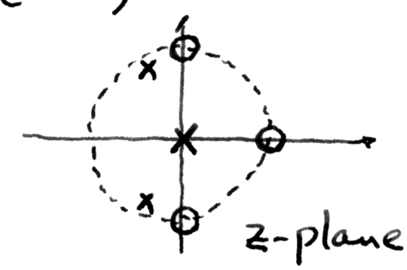
$$y[n] = 0.9y[n-1]-0.81y[n-2]+x[n]-x[n-1]+x[n-2]-x[n-3]$$

(b) Multiply numerator & denominator by $z^3$:

$$H(z) = \frac{(z-1)(z-j)(z+j)}{z(z-0.9e^{j2\pi/3})(z-0.9e^{-j2\pi/3})}$$

Zeroes: $z=1$, $j$ and $-j$

Poles: $z=0$, $z=0.9e^{\pm j2\pi/3}$




(c) The zeros of the numerator polynomial are on the unit circle at $z=e^{j0}$, $z=e^{j\pi/2}$ and $z=e^{-j\pi/2}$

When $x[n] = Ae^{j\varphi}e^{j\hat{\omega}n}$, the output $y[n]$ is

$$y[n] = H(e^{j\hat{\omega}})\cdot Ae^{j\varphi}e^{j\hat{\omega}n}$$

There the output will be zero when $H(e^{j\hat{\omega}})=0$.

That is, for $\hat{\omega}=0$, $\hat{\omega}=\pi/2$ and $\hat{\omega}=-\pi/2$.

# PROBLEM 8.20:

Using $f_s = 1000$ samples/sec, we can determine $x[n]$.

$$x[n] = x(t)\Big|_{t=n/f_s} = 4 + \cos\left(500\pi \frac{n}{1000}\right) - 3\cos\left(2000\frac{\pi}{3} \cdot \frac{n}{1000}\right)$$

$$x[n] = 4 + \cos\left(\frac{\pi}{2}n\right) - 3\cos\left(\frac{2\pi}{3}n\right)$$

Use the frequency response at $\hat{\omega} = 0, \frac{\pi}{2}$ and $\frac{2\pi}{3}$ to determine $y[n]$:

$$y[n] = 4H(e^{j0}) + |H(e^{j\pi/2})|\cos\left(\frac{\pi}{2}n + \angle H(e^{j\pi/2})\right)$$
$$-3|H(e^{j2\pi/3})|\cos\left(\frac{2\pi}{3}n + \angle H(e^{j2\pi/3})\right)$$

Since $H(z)$ has zeros at $z = 1$ and $z = e^{\pm j\pi/2}$, the frequency response is zero at $\hat{\omega} = 0$ & $\hat{\omega} = \pi/2$

Thus we only need $H(e^{j2\pi/3})$:

$$H(e^{j2\pi/3}) = \frac{(1 - e^{-j2\pi/3})(1 - e^{j\pi/2}e^{-j2\pi/3})(1 - e^{-j\pi/2}e^{-j2\pi/3})}{(1 - 0.9)(1 - 0.9e^{-j4\pi/3})}$$

$$= 10.522e^{j0.657\pi}$$

ANGLE = 118.26° or 2.064 rads

$$y[n] = -3(10.522)\cos\left(\frac{2\pi}{3}n + 0.657\pi\right)$$
$$= 31.566\cos\left(\frac{2\pi}{3}n - 0.343\pi\right)$$

INCORPORATE MINUS SIGN IN THE PHASE

Now convert back to continuous-time.

$$y(t) = y[n]\Big|_{n \leftarrow f_s t} = 31.566\cos\left(\frac{2\pi}{3}(1000)t - 0.343\pi\right)$$

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## PROBLEM 8.21:

(a) $$H(z) = \frac{1+0.8z^{-1}}{1-0.9z^{-1}} = \frac{z+0.8}{z-0.9}$$

$\therefore$ zero at $z = -0.8$

pole at $z = 0.9$

(b) $$y[n] = 0.9y[n-1] + x[n] + 0.8x[n-1]$$

(c) $$H(e^{j\hat{\omega}}) = \frac{1+0.8e^{-j\hat{\omega}}}{1-0.9e^{-j\hat{\omega}}}$$

CONJUGATE

$$|H(e^{j\hat{\omega}})|^2 = H(e^{j\hat{\omega}})H^*(e^{j\hat{\omega}})$$

$$= \frac{1+0.8e^{-j\hat{\omega}}}{1-0.9e^{-j\hat{\omega}}} \cdot \frac{1+0.8e^{+j\hat{\omega}}}{1-0.9e^{+j\hat{\omega}}}$$

$$= \frac{1.64+0.8e^{+j\hat{\omega}}+0.8e^{-j\hat{\omega}}}{1.81-0.9e^{j\hat{\omega}}-0.9e^{-j\hat{\omega}}}$$

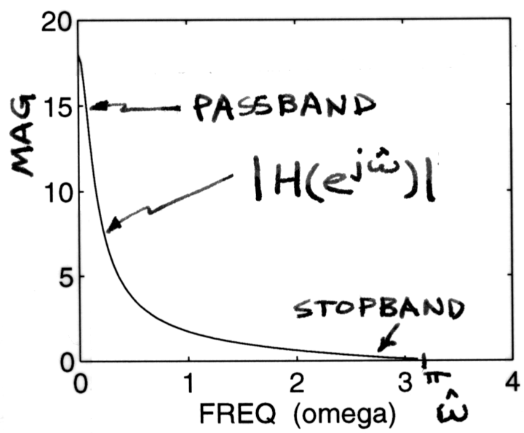
$$= \frac{1.64+1.6\cos\hat{\omega}}{1.81-1.8\cos\hat{\omega}} = \begin{cases} 324 & \hat{\omega}=0 \\ 0.0111 & \hat{\omega}=\pi \end{cases}$$

(d) It is a

LOWPASS

FILTER

see plot



## PROBLEM 8.22:

(a) $\left.\begin{aligned} Y_1(z) &= H_1(z)X(z) \\ Y(z) &= H_2(z)Y_1(z) \end{aligned}\right\} \Rightarrow \quad Y(z) = \underbrace{H_2(z)H_1(z)}_{H(z)}X(z)$

(b) $H_1(z) = 1 + \frac{5}{6}z^{-1}$

$$H(z) = H_2(z)H_1(z) = (1-2z^{-1}+z^{-2})(1+\tfrac{5}{6}z^{-1})$$

$$= 1 + \tfrac{5}{6}z^{-1} - 2z^{-1} - \tfrac{5}{3}z^{-2} + z^{-2} + \tfrac{5}{6}z^{-3}$$

$$= 1 - \tfrac{7}{6}z^{-1} - \tfrac{2}{3}z^{-2} + \tfrac{5}{6}z^{-3}$$

(c) Use polynomial coeffs as filter coefficients

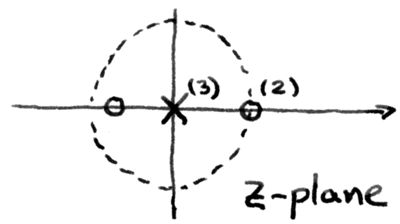
$$y[n] = x[n] - \tfrac{7}{6}x[n-1] - \tfrac{2}{3}x[n-2] + \tfrac{5}{6}x[n-3]$$

(d) $1 + \frac{5}{6}z^{-1} = \frac{z+5/6}{z} \Rightarrow$ zero at $z=-5/6$ pole at $z=0$

$$1-2z^{-1}+z^{-2} = \frac{z^2-2z+1}{z^2}$$

Two poles at $z=0$

2 Zeros at $z=+1$




(e) $y[n] = x[n] \Rightarrow Y(z) = X(z)$

$$\Rightarrow H(z) = 1$$

(f) $\left.\begin{aligned} H_1(z) &= 1 + 5/6\, z^{-1} \\ H(z) &= 1 \end{aligned}\right\} \Rightarrow \quad \begin{aligned} H(z) &= H_2(z)H_1(z) \\ 1 &= H_2(z)(1+5/6\,z^{-1}) \end{aligned}$

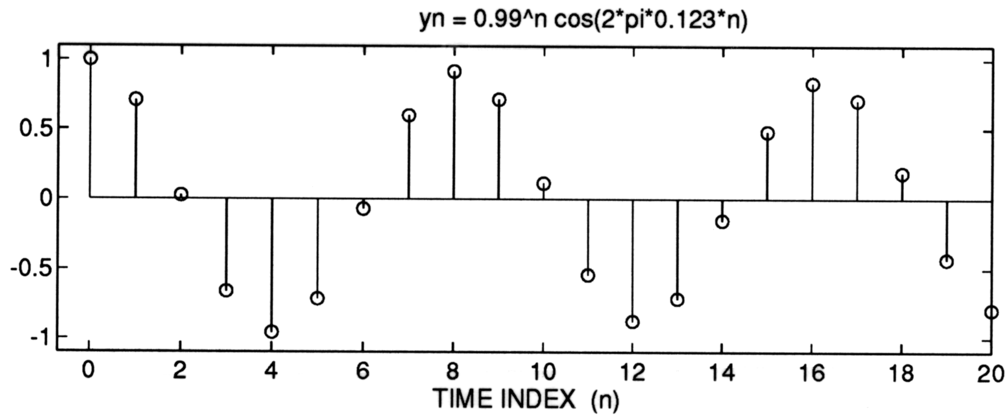
$$\rightarrow H_2(z) = \frac{1}{1+5/6\,z^{-1}}$$

Difference Eqn: $y[n] = -\frac{5}{6}y[n-1] + x[n]$

(g) $H_1(z)$ becomes the denominator polynomial in $H_2(z) = 1/H_1(z)$. Thus the zeros of $H_1(z)$ will have to be inside the unit circle if $H_2(z)$ is going to be a stable system.

## PROBLEM 8.23:

(a) Let $\varphi = 0$ & use MATLAB to make the plot




(b) To synthesize we need 2 poles at

$$z = 0.99\, e^{\pm j 2\pi(0.123)}$$

$$H(z) = \frac{1}{\left(1 - 0.99 e^{j2\pi(0.123)} z^{-1}\right)\left(1 - 0.99 e^{-j2\pi(0.123)} z^{-1}\right)}$$

$$= \frac{1}{1 - 1.98\cos(0.246\pi) z^{-1} + (0.99)^2 z^{-2}}$$

$$= \frac{1}{1 - 1.4176 z^{-1} + 0.9801 z^{-2}}$$

$$y[n] = +1.4176\, y[n-1] - 0.9801\, y[n-2] + x[n]$$

NOTE: this will not produce the $\varphi = 0$ case

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