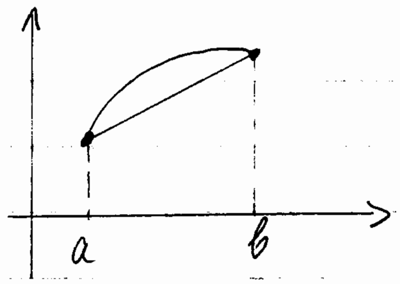
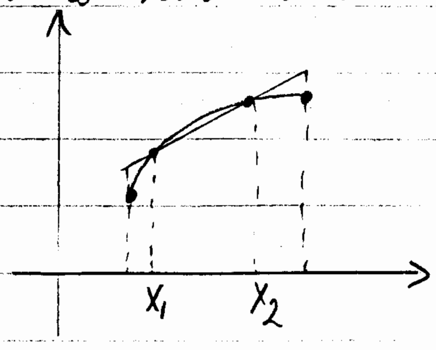
4.7 Gaussian Quadrature

The quadrature formulas in the previous section were obtained by approximating the integrand function with a polynomial and integrating the polynomial instead. Such formulas are called <u>Newton-Cotes formulas</u>.

All Newton-Cotes formulas use values of the function at equally spaced points.



In the trapezoidal rule we approximate with the line joining the values at the endpoints. This in most cases however this is not the best line.



Clearly approximating the integral with this line will be much accurate and possibly exact.

Gaussian quadrature choses the points for evaluation in an optimal (rather than equally spaced) way.

Problem: Choose the nodes

$x_1, \dots, x_n$ in $[a,b]$

and the coefficients

$c_1, \dots, c_n$

so that the error of the approximat.

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} c_i f(x_i)$$

be as small as possible.

Criterion for accuracy: We want to choose $x_1, \dots, x_n, c_1, \dots, c_n$ so that the formula is exact for the largest class of polynomials.

How many parameters to choose? $2n$

A formula is exact for ANY polynomia of degree n iff it is exact for

$1, x, x^2, \dots, x^n$

Since we have $2n$ parameters to choose, we can pose $2n$ conditions

Thus, we want the formula to be exact for

$$1, x, x^2, \dots, x^{2n-1}$$

Thus, a formula will be exact for all polynomials of degree $2n-1$.

Ex: Suppose we are approximating

$$\int_{-1}^{1} f(x)\,dx \approx c_1 f(x_1) + c_2 f(x_2)$$

and we want to choose c_1, c_2, x_1, x_2 so that the formula is exact (we have equality instead approximate equality) for the polynomials

$$1, x, x^2, x^3$$

Thus

$$\int_{-1}^{1} 1\,dx = c_1 + c_2 \implies c_1 + c_2 = 2$$

$$\int_{-1}^{1} x\,dx = c_1 x_1 + c_2 x_2 \implies c_1 x_1 + c_2 x_2 = 0$$

$$\int_{-1}^{1} x^2\,dx = c_1 x_1^2 + c_2 x_2^2 \implies c_1 x_1^2 + c_2 x_2^2 = \frac{2}{3}$$

$$\int_{-1}^{1} x^3\,dx = c_1 x_1^3 + c_2 x_2^3 \implies c_1 x_1^3 + c_2 x_2^3 = 0$$

Legendre polynomials $P_0(x), P_1(x), \dots, P_n(x), \dots$ where

$$\deg P_n(x) = n$$

have the property that

$$\int_{-1}^{1} Q(x) P_n(x)\,dx = 0$$

for every polynomial $Q(x)$ with

$$\deg Q(x) \le n-1$$

The first few Legendre polynomials are

$$P_0(x) = 1 \quad P_1(x) = x \quad P_2(x) = x^2 - \frac{1}{3}$$

$$P_3(x) = x^3 - \frac{3}{5}x \quad P_4(x) = x^4 - \frac{6}{7}x^2 + \frac{3}{35}$$

The roots of these polynomials are distinct and lie in $(-1, 1)$. They are taken as $x_1, \dots, x_n$.

It also turns out that $c_1, \dots, c_n$ can be determined

$$c_i = \int_{-1}^{1} L_i(x)\,dx$$

wher $L_i(x)$ is the basic Lagrange polynom.

Thus, we get a system for c_1, c_2, x_1, x_2

We get

$$c_1 = c_2 = 1 \quad x_1 = -\frac{\sqrt{3}}{3} \quad x_2 = \frac{\sqrt{3}}{3}$$

This gives the approximation formula

$$\int_{-1}^{1} f(x)\,dx = f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right)$$

This formula has a degree of precision 3, that is, the formula is exact for all polynomials of degree 3 or less.

Note: We don't need f to find $c_1, \dots, c_n, x_1, \dots, x_n$. It turns out that we can find $x_1, \dots, x_n$ and $c_1, \dots, c_n$ so that

$$\int_{-1}^{1} f(x)\,dx \approx c_1 f(x_1) + \dots + c_n f(x_n)$$

and the formula is exact for polynomials of degree $2n-1$ and this can be done (as in the example above) for any function f.

It turns out that the nodes

$$x_1, \dots, x_n$$

are the roots of the n^{th} Legendre polynomial.

$$L_i(x) = \frac{(x-x_1)\dots(x-x_{i-1})(x-x_{i+1})\dots(x-x_n)}{(x_i-x_1)\dots(x_i-x_{i-1})(x_i-x_{i+1})\dots(x_i-x_n)}$$

Actually, the values of $x_1, \dots, x_n, c_1, \dots, c_n$ are tabulated. See Table 4.11 (p. 225). p. 224

If we want to use Gauss quadrature to approximate

$$\int_a^b f(x)\,dx$$

over an arbitrary interval $[a,b]$ we first have to change variables so that the interval becomes $[-1,1]$.

$$t = \frac{2x-a-b}{b-a} \qquad x = \frac{1}{2}\left[(b-a)t + a + b\right]$$

Thus, we have

$$\int_a^b f(x)\,dx = \int_{-1}^{1} f\left(\frac{(b-a)t+a+b}{2}\right) \cdot \frac{b-a}{2}\,dt$$

Ex. Use Gaussian quadrature with $n=3$ to approximate the integral. Compare your result to the exact value of the integral

$$\int_0^{\frac{\pi}{4}} x^2 \sin x\,dx$$

First we change variables

$$x = \frac{1}{2}\left[\frac{\pi}{4}t + \frac{\pi}{4}\right] = \frac{\pi}{8}(t+1) \quad -1 \le t \le 1$$

$$\int_0^{\frac{\pi}{4}} x^2 \sin x\, dx = \int_{-1}^{1} \frac{\pi^2}{64}(t+1)^2 \sin \frac{\pi}{8}(t+1) \cdot \frac{\pi}{8}\, dt$$

$$= \frac{\pi^3}{512} \int_{-1}^{1} (t+1)^2 \sin \frac{\pi}{8}(t+1)\, dt$$

From table 4.11 we have

$t_1 = -0.7745966692 \quad t_2 = 0 \quad t_3 = 0.7745966692$

$c_1 = \frac{5}{9} \qquad c_2 = \frac{8}{9} \qquad c_3 = \frac{5}{9}$

$$\int_0^{\frac{\pi}{4}} x^2 \sin x\, dx \approx \frac{5}{9} \cdot \left(\frac{\pi}{8}\right)^3 (-0.7745966692+1)^2 \sin \frac{\pi}{8}(-0.7745+1)$$

$$+ \frac{8}{9} \cdot \frac{\pi^3}{512} \sin \frac{\pi}{8}$$

$$+ \frac{5}{9}\frac{\pi^3}{512} (0.7745966692+1)^2 \sin \frac{\pi}{8}(0.7745966692+1)$$

$$= 1.511056695 * 10^{-4} +$$

$$+ 0.0205999798$$

$$+ 0.0680027681 = 0.0887538536$$

The exact value is 0.088755284436

Error $\approx -1.43 \times 10^{-6}$

Ex. Determine the parameters a, b, c, d, e so that the formula

$$\int_0^2 f(x)\,dx \approx a f(0) + b f(1) + c f(2) + d f'(0) + e f'(2)$$

has highest order of precision

$$\int_0^2 1\,dx = a + b + c \qquad a+b+c = 2$$

$$\frac{x^2}{2}\Big|_0^2 = \int_0^2 x\,dx = b + 2c + d + e \qquad b+2c+d+e = 2$$

$$\frac{x^3}{3}\Big|_0^2 = \int_0^2 x^2\,dx = b + 4c + 4e \qquad b+4c+4e = \frac{8}{3}$$

$$\frac{x^4}{4}\Big|_0^2 = \int_0^2 x^3\,dx = b + 8c + 12e \qquad b+8c+12e = 4$$

$$\frac{x^5}{5}\Big|_0^2 = \int_0^2 x^4\,dx = b + 16c + 32e \qquad b+16c+32e = \frac{32}{5}$$

We get $a = \frac{7}{15}$ $b = \frac{16}{15}$ $c = \frac{7}{15}$ $d = \frac{1}{15}$ $e = -\frac{1}{15}$

Ex. Use Gaus quadrature to evaluate

$$\int_0^{\frac{\pi}{2}} \sin x \, dx \quad \text{with} \quad n=2$$

Then use Trapezoidal and Simpson's rule and compare the errors of 3 methods.

Solution: $x = \frac{1}{2}\left[\frac{\pi}{2}t + \frac{\pi}{2}\right] = \frac{\pi}{4}(t+1) \quad dx = \frac{\pi}{4}dt$

$$\int_0^{\frac{\pi}{2}} \sin x \, dx = \frac{\pi}{4}\int_{-1}^{1} \sin\frac{\pi}{4}(t+1)\, dt$$

From table 4.11 the nodes are

$t_1 = -0.57735 \qquad t_2 = 0.57735$

and the weights are

$c_1 = 1 \qquad c_2 = 1$

Thus,

$$\frac{\pi}{4}\int_{-1}^{1} \sin\frac{\pi}{4}(t+1)\,dt = \frac{\pi}{4}\left[1\cdot\sin\frac{\pi}{4}(-0.57735+1) + \right.$$

$$\left. + 1\sin\frac{\pi}{4}\left(0.57735+1\right)\right] = 0.99847$$

Exact value is

$$\int_0^{\frac{\pi}{2}} \sin x \, dx = -\cos x\Big|_0^{\frac{\pi}{2}} = -\cos\frac{\pi}{2} + \cos 0 = 1$$

Actual error $= 1.53 \cdot 10^{-3}$

$\int_a^b f(x)dx = \frac{h}{2}[f(x_0)+2f(x_1)+\dots+f(x_n)]$ $h = \frac{b-a}{2} = \frac{\frac{\pi}{2}}{2} = \frac{\pi}{4}$ ...

Using Trapezoidal rule we have ($n=2$)

$$\int_0^{\frac{\pi}{2}} \sin x\,dx = \frac{\pi}{8}\left(\sin 0 + 2\sin\frac{\pi}{4} + \sin\frac{\pi}{2}\right) = 0.948$$

Error $= 0.05194 = 5.194 \times 10^{-2}$

Using Simpson's rule we have ($h = \frac{\frac{\pi}{2}}{2} = \frac{\pi}{4}$)

$$\int_0^{\frac{\pi}{2}} \sin x\,dx = \frac{\pi}{12}\left[\sin 0 + 4\cdot\sin\frac{\pi}{4} + \sin\frac{\pi}{2}\right] =$$

$$= \frac{\pi}{12}\left[4\cdot\frac{\sqrt{2}}{2} + 1\right] = 1.00228$$

Actual error $= -0.00228 = -2.28 \cdot 10^{-3}$

Remark: Trapezoidal rule has two evaluations of f (as many as Gaussian method has) but gives much worse approximation. Simpson's rule gives approximately the same approximation (only slightly worse) but it has 3 evaluations of the function.

Conclusion: Gaussian quadrature are among the best