Ex. #11/64

For the equation

$$x = \left(\frac{e^x}{3}\right)^{\frac{1}{2}} = \frac{e^{\frac{x}{2}}}{\sqrt{3}}$$

determine an interval $[a,b]$ on which fixed-point iteration will converge. Estimate the number of iterations necessary to obtain approximations accurate within 10

$$g(x) = \frac{e^{\frac{x}{2}}}{\sqrt{3}}$$

1) We have to locate a root

$$f(x) = x - g(x)$$

$$f(0) = -\frac{1}{\sqrt{3}} < 0$$

$$f(1) = 1 - \frac{\sqrt{e}}{\sqrt{3}} = \frac{\sqrt{3}-\sqrt{e}}{\sqrt{3}} > 0$$

Thus, there is a root in $[0,1]$.

2) Note

$$\frac{1}{\sqrt{3}} \le g(x) \le 1 \quad \forall\, x \in [0,1]$$

3) $g'(x) = \frac{e^{\frac{x}{2}}}{2\sqrt{3}} \Rightarrow |g'(x)| \le \frac{1}{2}$

increasing

Lets take $p_0 = 0$ $p_1 = \frac{1}{\sqrt{3}}$

We use the formula $|p_n - p| \le \frac{k^n}{1-k}|p_1 - p_0|$

$$|p_n - p| \le \frac{\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} \cdot \frac{1}{\sqrt{3}} = \frac{2}{\sqrt{3}}\left(\frac{1}{2}\right)^n < 10^{-5}$$