5.3 Higher Order Taylor Methods

1) Local truncation errors

We need a way to compare the efficiency of various approximation methods. The first device is called „local truncation error".

To illustrate the concept, recall Euler's method. We obtained the method by expanding $y(t_{i+1})$ in Taylor's series around t_i.

$$y(t_{i+1}) = y(t_i) + h y'(t_i) + \frac{h^2}{2!} y''(\xi_i)$$

$$\xi \text{ in } (t_i, t_{i+1})$$

Then we ignored the term $\frac{h^2}{2!} y''(\xi_i)$.

So if the values of $y(t_i)$, $y'(t_i)$ are exact the error we will make computing $y(t_{i+1})$ by Euler's method is

$$\frac{h^2}{2!} y''(\xi_i)$$

It is customary to call this error divided by h the <u>local truncation error</u>

$$\tau_{i+1}(h) = \frac{h}{2!} y''(\xi_i) \qquad \xi_i \text{ in } (t_i, t_{i+1})$$

local truncation error

If $|y''(\xi)| \leq M$ on $[a,b]$ then

$$|\tau_{i+1}(h)| \leq \frac{M}{2} h$$

Thus, the local truncation error of Euler's method is $O(h)$.

The local truncation error is <u>local</u> because it measures the accuracy at a specific step, assuming that the method has produced exact values at the previous step.

2) Higher Order Taylor's methods

Euler's method was derived from Taylor's expansion stopping at the second derivative. We can derive more methods if we stop at higher order derivatives

We are solving the IVP

$$\begin{cases} y' = f(t,y) & a \le t \le b \\ y(a) = \alpha \end{cases}$$

Assume the solution $y(t)$ has $(n+1)$ continuous derivatives. Then

$$y(t_{i+1}) = y(t_i) + h y'(t_i) + \frac{h^2}{2!} y''(t_i) + \dots + \frac{h^n}{n!} y^{(n)}(t_i) + \frac{h^{n+1}}{(n+1)!} y^{(n+1)}(\xi_i)$$

ξ_i in (t_i, t_{i+1})

Problem: we need $y''(t_i), \dots, y^{(n)}(t_i)$
We can get them by successively differentiating the equation

$$y'(t) = f(t, y(t))$$

$$y''(t) = \frac{d}{dt} f(t, y(t)) = f'(t, y(t))$$

$$\dots$$

$$y^{(k)}(t) = f^{(k-1)}(t, y(t))$$

Thus, we have:

$$y(t_{i+1}) = y(t_i) + h f(t_i, y(t_i)) + \frac{h^2}{2!} f'(t_i, y(t_i)) + \dots + \frac{h^n}{n!} f^{(n-1)}(t_i, y(t_i)) + \underbrace{\frac{h^{n+1}}{(n+1)!} f^{(n)}(\xi_i, y(\xi_i))}_{\text{drop}}$$

Thus, we define Taylor method of order n:

$$w_0 = \alpha$$
$$w_{i+1} = w_i + h\, T^{(n)}(t_i, w_i) \quad i = 0, \dots, N-1$$

where

$$T^{(n)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2!} f'(t_i, w_i) + \dots + \frac{h^{n-1}}{n!} f^{(n-1)}(t_i, w_i)$$

Note: Euler's method is Taylor's method of order 1.

The term that we dropped was

$$\frac{h^{n+1}}{(n+1)!} f^{(n)}(\xi_i, y(\xi_i))$$

Thus, the local truncation error of Taylor method of order n is

$$\tau_{i+1}(h) = \frac{h^n}{(n+1)!} f^{(n)}(\xi_i, y(\xi_i))$$

Thus, if

$$|f^{(n)}(t, y(t))| \le M \quad \text{for } t \text{ in } [a, b]$$

then

$$|\tau_{i+1}(h)| \le \frac{M}{(n+1)!} h^n$$

Thus, the local truncation error is $O(h^n)$

3) Examples

Ex: Compose Taylor's methods of order 2 and 4 to solve the following IVP

(#1ª, 2ª/271)

$$y' = te^{3t} - 2y \qquad 0 \le t \le 1$$
$$y(0) = 0$$

with $h = 0.5$

Thus, we have

$$y' = te^{3t} - 2y = f(t,y)$$

Composing Taylor's method of order 2 requires

$$y'' = f'(t,y)$$

$$f'(t,y) = e^{3t} + 3te^{3t} - 2y' =$$

$$= e^{3t} + 3te^{3t} - 2(te^{3t} - 2y)$$
$$= e^{3t} + 3te^{3t} - 2te^{3t} + 4y$$
$$= e^{3t} + te^{3t} + 4y$$
$$= (1+t)e^{3t} + 4y$$

Composing Taylor's method of order 4 requires $y''' = f''(t,y)$, $y^{iv} = f'''(t,y)$. Thus, we continue differentiating $f(t,y)$

$$f''(t,y) = e^{3t} + 3(1+t)e^{3t} + 4y'$$
$$= (1+3+3t)e^{3t} + 4(te^{3t} - 2y)$$
$$= (4+7t)e^{3t} - 8y$$

$$f'''(t,y) = 7e^{3t} + 3(4+7t)e^{3t} - 8y'$$
$$= (7+12+21t)e^{3t} - 8(te^{3t} - 2y)$$
$$= (19+13t)e^{3t} + 16y$$

Taylor's method of order 2

$$w_0 = \alpha$$
$$w_{i+1} = w_i + h\, T^{(2)}(t_i, w_i)$$

where

$$T^{(2)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i)$$
$$= t_i e^{3t_i} - 2w_i + \frac{h}{2}\left[(1+t_i)e^{3t_i} + 4w_i\right]$$
$$= \left(1+\frac{h}{2}\right) t_i e^{3t_i} + \frac{h}{2} e^{3t_i} - 2(1-h)w_i$$

Thus, for this example we have

$$\begin{cases} w_0 = 0 \\ w_{i+1} = w_i + 0.5\left[1.25\, t_i e^{3t_i} + 0.25 e^{3t_i} - w_i\right] \end{cases}$$
$$= 0.5 w_i + 0.625\, t_i e^{3t_i} + 0.125 e^{3t_i}$$

Taylor's method of order 4:

$$\begin{cases} w_0 = \alpha \\ w_{i+1} = w_i + h\,T^{(4)}(t_i, w_i) \end{cases}$$

where

$$T^{(4)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) + \frac{h^2}{6} f''(t_i, w_i)$$

$$+ \frac{h^3}{24} f'''(t_i, w_i) =$$

$$= t_i e^{3t_i} - 2w_i + \frac{h}{2}\left((1+t_i)e^{3t_i} + 4w_i\right)$$

$$+ \frac{h^2}{6}\left((4+7t_i)e^{3t_i} - 8w_i\right)$$

$$+ \frac{h^3}{24}\left((19+13t_i)e^{3t_i} + 16w_i\right)$$

$$= \left(1 + \frac{h}{2} + \frac{7}{6}h^2 + \frac{13}{24}h^3\right) t_i e^{3t_i} + \left(\frac{h}{2} + \frac{2h^2}{3} + \frac{19h^3}{24}\right) e^{3t_i}$$

$$+ \left(-2 + 2h - \frac{4}{3}h^2 + \frac{2}{3}h^3\right) w_i$$

For $h = 0.5$ we get

$$= 1.609375 t_i e^{3t_i} + 0.515625 e^{3t_i} - 1.25 w_i$$

Thus,

$$\begin{cases} w_0 = 0 \\ w_{i+1} = 0.375 w_i + 0.8046875 t_i e^{3t_i} \\ \qquad + 0.2578125 e^{3t_i} \end{cases}$$