PROBLEM 4.1:

(a) $x[n] = x(nT_s) = 10\cos(880\pi n T_s + \varphi)$ $\quad T_s = 0.0001$

$$880T_s = 880 \times 10^{-4} = 0.088 = 11/125$$

To find the number of samples within one period of the continuous cosine $x(t)$, find the largest integer satisfying $880\pi n T_s \le 2\pi$

$$n \le \frac{2}{0.088} = \frac{250}{11} = 22.73$$

There are 23 samples in one period, because samples $n = 0, 1, 2, \ldots 22$ are within one period.

NOTE: the period of $x[n]$ is <u>not</u> 23; it is actually 250.

(b) $y[n] = 10\cos(\omega_0 n T_s + \varphi)$

To get the same samples for $x[n]$ & $y[n]$ we solve:

$$\omega_0 n T_s = 880\pi n T_s + 2\pi \ell n \qquad \ell = \text{integer}$$

$$\Rightarrow \omega_0 = 880\pi + \frac{2\pi\ell}{T_s} \qquad \frac{2\pi}{T_s} = 20{,}000\pi$$

Take $\ell = 1$: $\omega_0 = 20{,}880\pi$

(c) Find largest integer satisfying

$$(20{,}880\pi)\, n T_s \le 2\pi$$

$$n \le \frac{2}{2.088} \quad \text{which is less than one!}$$

$\therefore$ only one sample per period is taken

McClellan, Schafer and Yoder, *Signal Processing First*, ISBN 0-13-065562-7.
Pearson Prentice Hall, Inc. Upper Saddle River, NJ 07458. © 2003

PROBLEM 4.2:

$$x(t) = 7\sin(11\pi t) = 7\cos(11\pi t - \pi/2) \longrightarrow \boxed{A/D} \longrightarrow x[n] = A\cos(\hat{\omega}_0 n + \varphi).$$

$$f_s.$$

(a) $f_s = 10$ samples/sec.

$$x(t)\Big|_{t=n/f_s} = x\left(\frac{n}{f_s}\right) = 7\cos\left(\frac{11\pi n}{10} - \pi/2\right).$$

$$= 7\cos\left(\frac{11\pi n}{10} - 2\pi n - \pi/2\right).$$

$$= 7\cos\left(-\frac{9\pi n}{10} - \pi/2\right) = 7\cos(.9\pi n + \pi/2).$$

$$\boxed{A = 7,\ \hat{\omega}_0 = 0.9\pi,\ \varphi = \pi/2}$$

(b) $f_s = 5$ samples/sec

$$x(t)\Big|_{t=n/f_s} = x\left(\frac{n}{5}\right) = 7\cos\left(\frac{11\pi n}{5} - \pi/2\right)$$

$$= 7\cos\left(\frac{\pi n}{5} - \frac{\pi}{2}\right)$$

$$\boxed{A = 7,\ \hat{\omega}_0 = \frac{\pi}{5},\ \varphi = -\frac{\pi}{2}}$$

(c) $f_s = 15$ samples/sec

$$x(t)\Big|_{t=n/f_s} = x\left(\frac{n}{15}\right) = 7\cos\left(\frac{11\pi n}{15} - \frac{\pi}{2}\right)$$

$$A = 7,\ \hat{\omega}_0 = \frac{11\pi}{15} = 2\pi\left(\frac{5.5}{15}\right)\ \&\ \varphi = -\pi/2$$

PROBLEM 4.3:

$x[n] = 2.2\cos(0.3\pi n - \pi/3)$ $\qquad \boxed{f_s = 6000}$

Compare to

$x(\frac{n}{f_s}) = A\cos(2\pi f_0 \frac{n}{f_s} + \varphi)$ ← sampled continuous-time signal.

$\Rightarrow \frac{2\pi f_0}{f_s} = 0.3\pi$, or $0.3\pi + 2\pi$, or $0.3\pi - 2\pi$.

Solve: $\frac{2\pi f_0}{f_s} = 0.3\pi \Rightarrow f_0 = f_s\left(\frac{0.3}{2}\right) = 6000 \times 0.15$

$f_0 = 900$ Hz

$\rightarrow x(t) = 2.2\cos(1800\pi t - \pi/3)$

NOTE: difference is f_s

Then $\frac{2\pi f_0}{f_s} = 2.3\pi \Rightarrow f_0 = f_s\left(\frac{2.3}{2}\right) = 6900$ Hz

$\longrightarrow x(t) = 2.2\cos(2\pi(6900)t - \pi/3)$.

Finally,

$\frac{2\pi f_0}{f_s} = -1.7\pi \Rightarrow f_0 = f_s\left(-\frac{1.7}{2}\right) = -5100$ Hz.

$x(t) = 2.2\cos(2\pi(-5100)t - \pi/3)$

$\rightarrow x(t) = 2.2\cos(2\pi(5100)t + \pi/3)$

McClellan, Schafer and Yoder, *Signal Processing First*, ISBN 0-13-065562-7.
Pearson Prentice Hall, Inc. Upper Saddle River, NJ 07458. © 2003

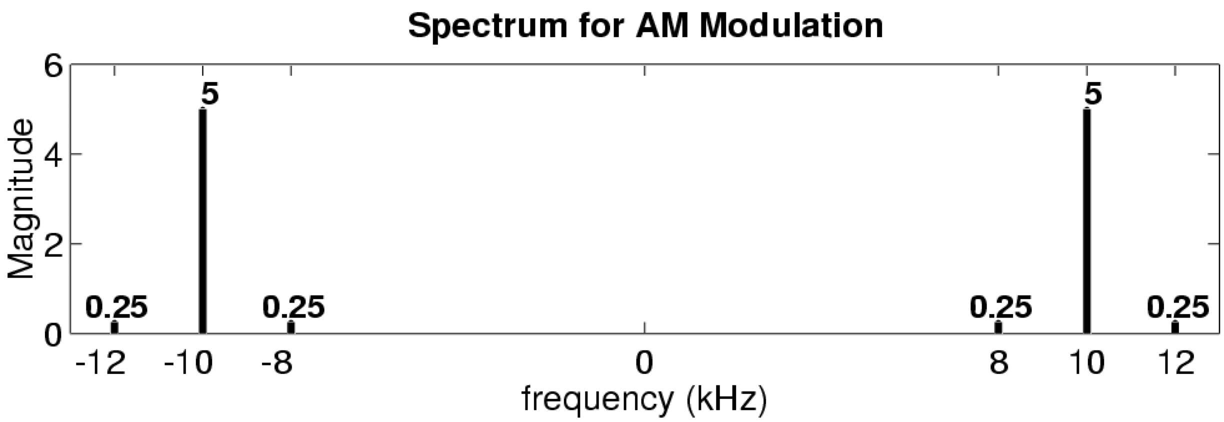
PROBLEM 4.4:

(a) $x(t) = \left[10 + \frac{1}{2}e^{j2\pi(2000)t} + \frac{1}{2}e^{-j2\pi(2000)t}\right]\left(\frac{1}{2}e^{j2\pi\times10^4 t} + \frac{1}{2}e^{-j2\pi\times10^4 t}\right)$

There are six terms:

$$x(t) = 5e^{j2\pi\times10^4 t} + 5e^{-j2\pi\times10^4 t} + \frac{1}{4}e^{j2\pi(12000)t} + \frac{1}{4}e^{-j2\pi(12000)t} + \frac{1}{4}e^{j2\pi(8000)t} + \frac{1}{4}e^{-j2\pi(8000)t}$$

Spectrum plot was created in MATLAB:



(b) Yes the waveform is periodic. The six frequencies $\{-12{,}000, -10{,}000, -8000, 8000, 10000, 12000\}$ are all divisible by 2000 Hz. Therefore, $f_0 = 2000$ Hz is the fundamental frequency. The period is $1/f_0 = 1/2000$ sec $= \frac{1}{2}$ msec

(c) The sampling rate must be greater than twice the highest frequency in $x(t)$.

$$\Rightarrow \quad f_s > 2(12{,}000) = 24{,}000 \text{ Hz}$$

McClellan, Schafer and Yoder, *Signal Processing First*, ISBN 0-13-065562-7.
Pearson Prentice Hall, Inc. Upper Saddle River, NJ 07458. © 2003

PROBLEM 4.5:

(a) Let $x(t) = 10\cos(\omega_0 t + \varphi)$

Sampling at a rate of $f_s \Rightarrow x[n] = x(t)\big|_{t=n/f_s} = x(n/f_s)$

$f_s = 1000$

$$x[n] = 10\cos\left(\omega_0 \frac{n}{f_s} + \varphi\right)$$

$\frac{\omega_0}{f_s} = 0.2\pi \Rightarrow \omega_0 = 0.2\pi \times 1000 = 200\pi$

Equate this to

$$x[n] = 10\cos(0.2\pi n - \pi/7)$$

$\varphi = -\pi/7$

A second possible signal is the "folded alias" at $(f_s - f_0)$

$$f_s - f_0 = f_s - \frac{\omega_0}{2\pi} = 1000 - \frac{200\pi}{2\pi} = 900\text{ Hz}$$

In this case, the phase (φ) changes.

$$\tilde{x}(t) = 10\cos(2\pi(f_s - f_0)t + \psi)$$

$$\tilde{x}[n] = 10\cos\left(2\pi(f_s - f_0)\frac{n}{f_s} + \psi\right) = 10\cos\left(2\pi n - 2\pi f_0 \frac{n}{f_s} + \psi\right)$$

$$= 10\cos\left(-\frac{2\pi f_0 n}{f_s} + \psi\right) = 10\cos\left(2\pi\frac{f_0}{f_s}n - \psi\right).$$

f_0 is still 100 Hz

$$\Rightarrow \psi = +\pi/7$$

(b) Reconstruction of $x[n]$ with $f_s = 2000$ samples/sec.

The discrete and continuous domains are related by: $\frac{n}{f_s} \longleftrightarrow t$ or $n \longleftrightarrow f_s t$

So we replace "n" in $x[n]$ with $f_s t$. This is what an <u>ideal</u> D-to-A would do.

$$x[n] = 10\cos(0.2\pi n - \pi/7)$$

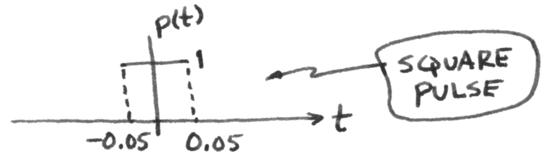
$$x(t) = 10\cos(0.2\pi f_s t - \pi/7) \quad \leftarrow f_s = 2000$$

$$= 10\cos(400\pi t - \pi/7)$$

$\omega_0 = 400\pi \Rightarrow f_0 = 200\text{ Hz}.$

PROBLEM 4.6:

(a) $p(t) = \begin{cases} 1 & -0.05 \le t \le 0.05 \\ 0 & \text{otherwise} \end{cases}$

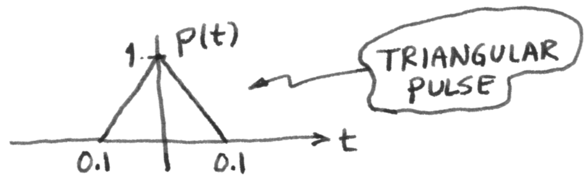


In the formula for $y(t)$

$$y(t) = \ldots + y[0]\,p(t) + y[1]\,p(t-T_s) + y[2]\,p(t-2T_s) + \ldots$$

The square pulses will <u>not</u> overlap, so the values of $y[n]$ will be extended over an interval of T_s.

(b) $p(t) = \begin{cases} 1 - 10|t| & -0.1 \le t \le 0.1 \\ 0 & \text{otherwise} \end{cases}$

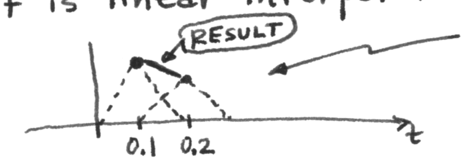


In this case, the neighboring terms do overlap

$$y(t) = \ldots + y[0]\,p(t) + y[1]\,p(t-T_s) + y[2]\,p(t-2T_s) + \ldots$$

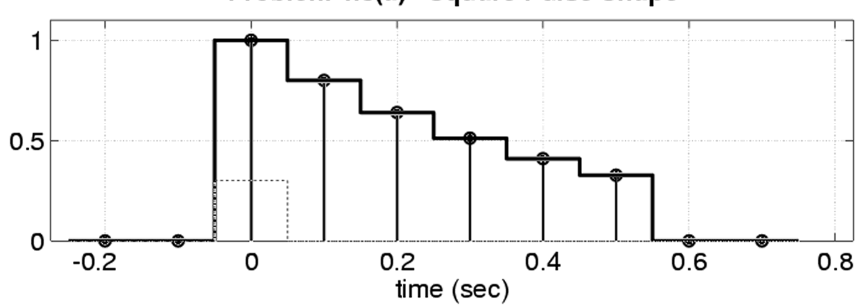
The result is linear interpolation.

Example:

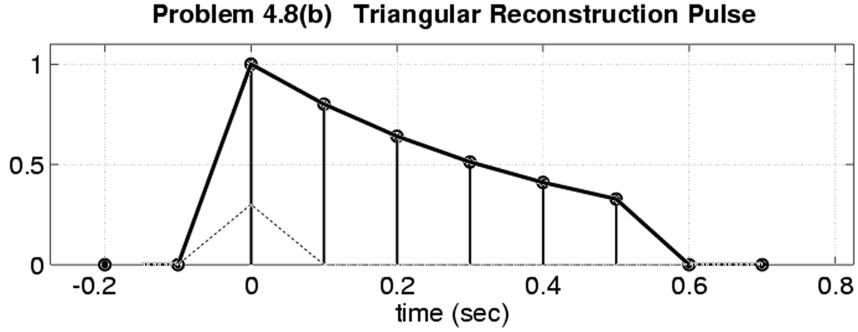


When we add these two triangles, the result between $t=0.1$ and $t=0.2$ is a straight line.

Problem 4.8(a) Square Pulse Shape



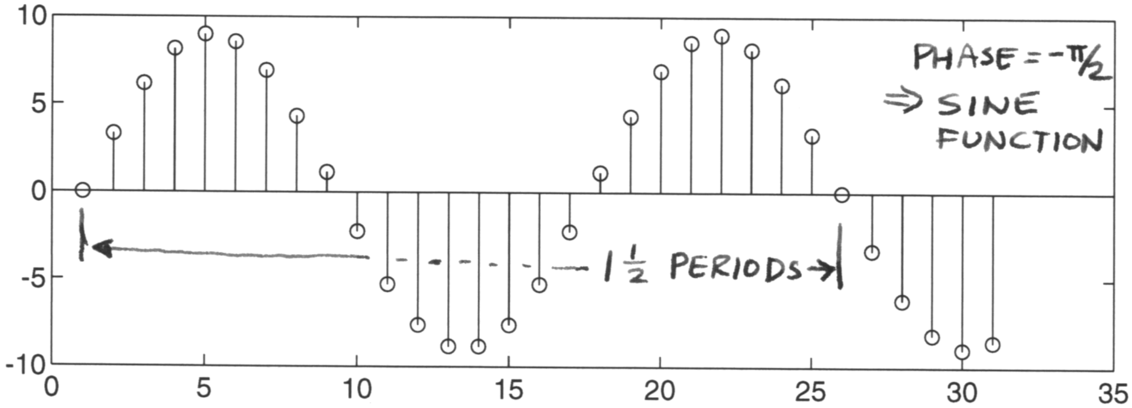
Problem 4.8(b) Triangular Reconstruction Pulse



McClellan, Schafer and Yoder, *Signal Processing First*, ISBN 0-13-065562-7.
Pearson Prentice Hall, Inc. Upper Saddle River, NJ 07458. © 2003

PROBLEM 4.7:

(a)



NOTE $1\frac{1}{2}$ PERIODS = 25 samples.

$\Rightarrow$ PERIOD = $50/3$ $\Rightarrow$ $\hat{\omega}_0 = 2\pi(0.06)$

$1/(50/3) = \frac{3}{50}$

(b) The vector xx is actually $x[n]$

$$x[n] = 9\cos(2\pi(394)n(0.01) + \pi/2)$$

$$= 9\cos(2\pi(3.94)n + \pi/2)$$

$$= 9\cos(2\pi(0.94)n + \pi/2)$$

REMOVED MULTIPLE OF 2π

$$= 9\cos(2\pi(-0.06)n + \pi/2)$$

$$x[n] = 9\cos(2\pi(0.06)n - \pi/2)$$

(c) Derivation in part (b) shows that $x[n]$ looks like samples of a 6 Hz sinusoid taken at $f_s = \frac{1}{0.01} = 100$ Hz.

McClellan, Schafer and Yoder, *Signal Processing First*, ISBN 0-13-065562-7.
Pearson Prentice Hall, Inc. Upper Saddle River, NJ 07458. © 2003

PROBLEM 4.8:

$$x(t) = \cos(50\pi t)\sin(700\pi t)$$

(a)

$$x(t) = \left(\tfrac{1}{2}e^{j50\pi t} + \tfrac{1}{2}e^{-j50\pi t}\right)\left(\tfrac{1}{2j}e^{j700\pi t} - \tfrac{1}{2j}e^{-j700\pi t}\right)$$

$$= \frac{1}{4j}e^{j750\pi t} + \frac{1}{4j}e^{j650\pi t} - \frac{1}{4j}e^{-j650\pi t} - \frac{1}{4j}e^{-j750\pi t}$$

SAME AS $\frac{1}{4}e^{-j\pi/2}$ SAME AS $\frac{1}{4}e^{+j\pi/2}$

SPECTRUM. SAME AS 650π rad/sec

(b) Sampling Thm says sample at a rate greater than $\boxed{\text{two}}\,\boxed{\text{times}}$ the highest freq.

HIGHEST FREQ = 375 Hz

$$\Rightarrow f_s \geq 750\text{ Hz}.$$

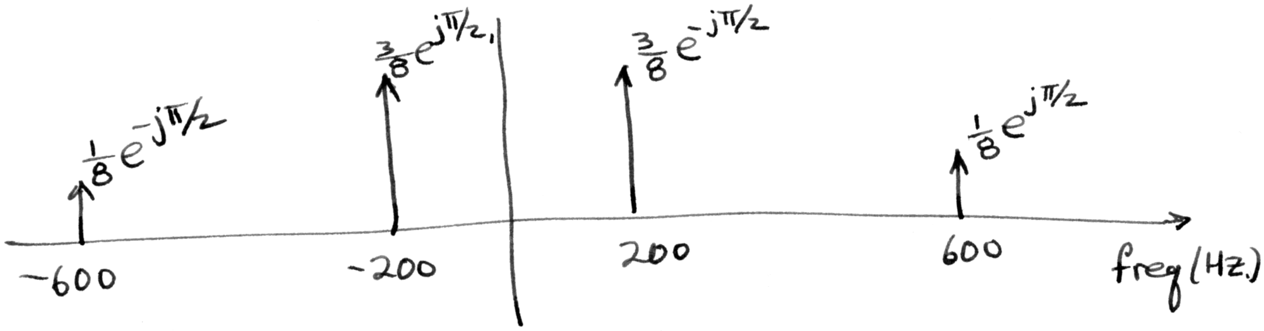
McClellan, Schafer and Yoder, *Signal Processing First*, ISBN 0-13-065562-7.
Pearson Prentice Hall, Inc. Upper Saddle River, NJ 07458.

PROBLEM 4.9:

(a) Draw a sketch of the spectrum of $x(t)$ which is "sine-cubed" $\quad x(t) = \sin^3(400\pi t)$

$$x(t) = \left(\frac{e^{j400\pi t} - e^{-j400\pi t}}{2j} \right)^3$$

$$= \frac{1}{-8j} \left\{ e^{j1200\pi t} - 3e^{j400\pi t} + 3e^{-j400\pi t} - e^{-j1200\pi t} \right).$$



$$\frac{1}{-8j} = \frac{1}{8} e^{j\pi/2}$$

(b) Determine the minimum sampling rate that can be used to sample $x(t)$ without any aliasing.

$$f_s \geq 2 f_{HIGH}$$

$$\Rightarrow f_s \geq 1200 \text{ Hz}$$

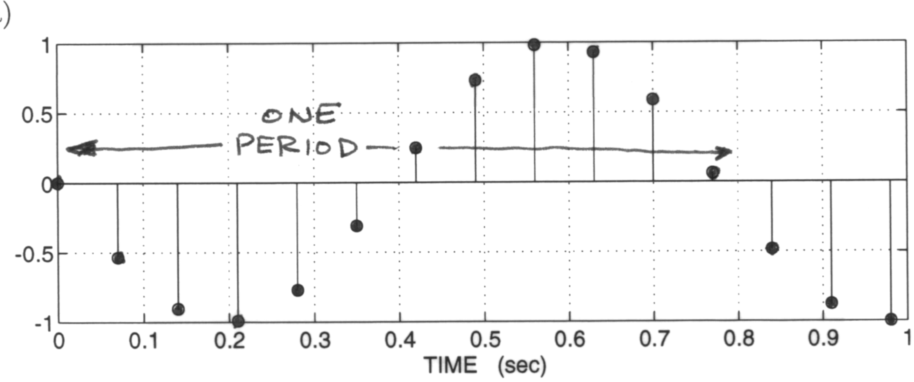
PROBLEM 4.10:

```
Fo = 13;
Period = 1/Fo;
Ts  = 0.07;
tt = 0 : Ts  : (13*Period);
j = sqrt(-1);
xx = real( exp( j*(2*pi*Fo*tt  - pi/2) ) );
%
stem( tt, xx ),  xlabel('TIME  (sec)'),  grid
```

$F_{SAMP} = 1/T_s = \frac{1}{0.07} = 14.28\text{Hz}.$

↑ NOT GREATER THAN $2F_0$

(a)



we observe 1 sec of the signal which is 1.28 periods

$$x[n] = \cos(2\pi(13)(0.07n) - \pi/2)$$
$$= \cos(2\pi(0.91)n - \pi/2)$$
$$= \cos(2\pi(0.09)n + \pi/2)$$

FOLDING

in continuous-time, the folded frequency is $14.28 - 13 = 1.28$ Hz

$\Rightarrow$ period $\approx \frac{1}{1.28} \approx .8$ sec

(b) SAMPLING THM $\Rightarrow$ $F_{SAMP} \geq 2F_0 = 2(13) = 26$

$F_{SAMP} = 1/T_s$

To get $\boxed{\text{SMOOTH}}$ plot need about 20 samples per period, which is a sampling rate of $20F_0$

$\therefore \quad T_s \leq \frac{1}{20F_0} = \frac{1}{20(13)} = \frac{1}{260}.$

McClellan, Schafer and Yoder, *Signal Processing First*, ISBN 0-13-065562-7.
Pearson Prentice Hall, Inc. Upper Saddle River, NJ 07458. © 2003

PROBLEM 4.11:

$$x(t) = [3 + \sin(\pi t)]\cos(13\pi t + \pi/2)$$

$\cos(\pi t - \pi/2)$

(a) Use *phasors* to show that $x(t)$ can be expressed in the form:

$$x(t) = A_1\cos(\omega_1 t + \phi_1) + A_2\cos(\omega_2 t + \phi_2) + A_3\cos(\omega_3 t + \phi_3)$$

where $\omega_1 < \omega_2 < \omega_3$; i.e., find $A_1,\ A_2,\ A_3,\ \phi_1,\ \phi_2,\ \phi_3,\ \omega_1,\ \omega_2,\ \omega_3$ in terms of A, ω_0, and ω_c.

$$x(t) = \left[3 + \tfrac{1}{2}e^{j(\pi t - \pi/2)} + \tfrac{1}{2}e^{-j(\pi t - \pi/2)}\right]\left(\tfrac{1}{2}e^{j(13\pi t + \pi/2)} + \tfrac{1}{2}e^{-j(13\pi t + \pi/2)}\right)$$

$$= \tfrac{3}{2}e^{j\pi/2}e^{j13\pi t} + \tfrac{3}{2}e^{-j\pi/2}e^{-j13\pi t} + \tfrac{1}{4}e^{j14\pi t} + \tfrac{1}{4}e^{-j14\pi t}$$

$$+ \tfrac{1}{4}e^{j\pi}e^{j12\pi t} + \tfrac{1}{4}e^{-j\pi}e^{-j12\pi t}$$

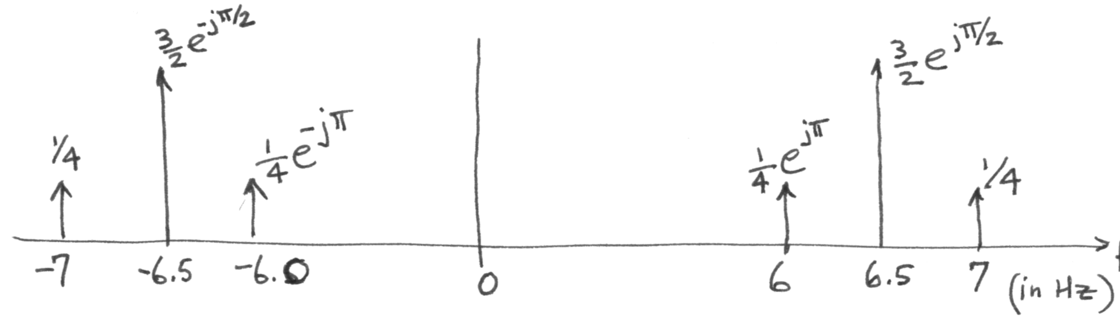
$$= 3\cos(13\pi t + \pi/2) + \tfrac{1}{2}\cos 14\pi t + \tfrac{1}{2}\cos(12\pi t + \pi)$$

$\omega_1 = 12\pi \qquad A_1 = 1/2 \qquad \phi_1 = \pi$

$\omega_2 = 13\pi \qquad A_2 = 3 \qquad \phi_2 = \pi/2$

$\omega_3 = 14\pi \qquad A_3 = 1/2 \qquad \phi_3 = 0$

(b) Sketch the two-sided spectrum of this signal on a frequency axis. Be sure to label important features of the plot. Label your plot in terms of the numerical values of the A_i's ϕ_i's and ω_i's.



(c) Determine the minimum sampling rate that can be used to sample $x(t)$ without any aliasing.

HIGHEST FREQ = 7 Hz.

$\Rightarrow F_{SAMP} \geq 2(7) = 14$ Hz

McClellan, Schafer and Yoder, *Signal Processing First*, ISBN 0-13-065562-7.
Pearson Prentice Hall, Inc. Upper Saddle River, NJ 07458. © 2003

PROBLEM 4.12:

(a) $x[n] = 10\cos(0.13\pi n + \pi/13)$ the sampling rate is $f_s = 1000$ samples/second

$.13\pi n = 2\pi(0.065)n = 2\pi(65)\frac{n}{1000}$ => 65 Hz is one Freq.

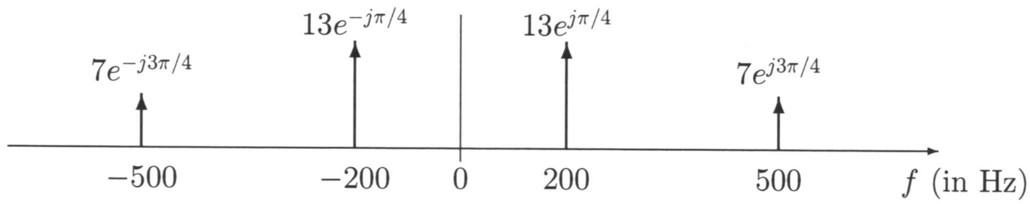
$x_1(t) = 10\cos(2\pi(65)t + \pi/13)$

Also, it could be "folded" case: $1000 - 65 = 935$ Hz

$x_2(t) = 10\cos(2\pi(935)t - \pi/13)$

NOTE phase reversal

(b) If the input $x(t)$ is given by the two-sided spectrum representation shown below, determine a $\boxed{\text{simple}}$ formula for $y(t)$ when $f_s = 700$ samples/sec. (for both the C/D and D/C converters).



$x(t) = 26\cos(2\pi(200)t + \pi/4) + 14\cos(2\pi(500)t + 3\pi/4)$

$x[n] = 26\cos(2\pi(\frac{2}{7})n + \pi/4) + 14\cos(2\pi(\frac{5}{7})n + 3\pi/4)$

THIS TERM FOLDS

$x[n] = 26\cos(2\pi(\frac{2}{7})n + \pi/4) + 14\cos(2\pi(\frac{-2}{7})n + 3\pi/4)$.

$\Rightarrow y(t) = 26\cos(2\pi(200)t + \pi/4) + 14\cos(2\pi(200)t - 3\pi/4)$.

$y(t) = 12\cos(2\pi(200)t + \pi/4)$

PROBLEM 4.13:

Assume that the sampling rates of a C-to-D and D-to-C conversion system are equal, and the input to the Ideal C-to-D converter is

$$x(t) = 2\cos(2\pi(50)t + \pi/2) + \cos(2\pi(150)t)$$

(a) If the output of the ideal D-to-C Converter is equal to the input $x(t)$, i.e.,

$$y(t) = 2\cos(2\pi(50)t + \pi/2) + \cos(2\pi(150)t)$$

what general statement can you make about the sampling frequency f_s in this case?

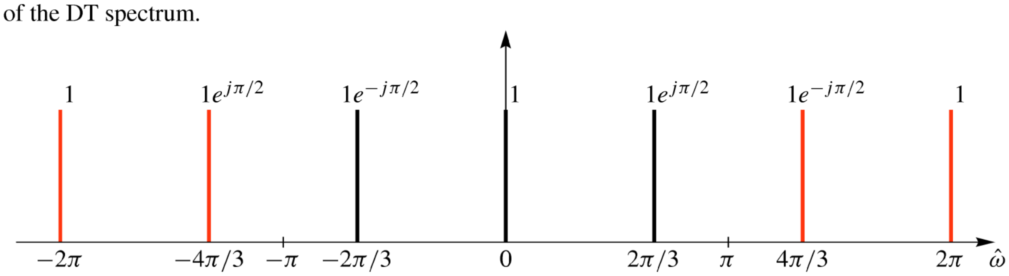
Solution: The sampling frequency must be greater than twice the highest frequency, because there was no aliasing. Thus, we can say that

$$F_s > 2 \times 150 = 300 \text{ Hz}$$

(b) If the sampling rate is $f_s = 250$ samples/sec., determine the discrete-time signal $x[n]$, and give an expression for $x[n]$ as a sum of cosines. *Make sure that all frequencies in your answer are positive and less than π radians.* *Solution:* Replace t with $n/f_s = n/250$ to get

$$\begin{aligned} x[n] = x(n/250) &= 2\cos(2\pi(50)(n/250) + \pi/2) + \cos(2\pi(150)(n/250)) \\ &= 2\cos(2\pi(0.2)n + \pi/2) + \cos(2\pi(0.6)n) \\ &= 2\cos(2\pi(0.2)n + \pi/2) + \cos(2\pi(0.4)n) \end{aligned}$$

(c) Plot the spectrum of the signal in part (b) over the range of frequencies $-\pi \le \hat{\omega} \le \pi$. The plot below shows the periodicity of the DT spectrum.



(d) If the output of the Ideal D-to-C Converter is

$$y(t) = 2\cos(2\pi(50)t + \pi/2) + 1$$

determine the value of the sampling frequency f_s. (Remember that the input signal is $x(t)$ defined above.)

Solution: Since the frequency of 50 Hz is preserved, the other frequency of 150 Hz must have been aliased to 0 Hz. This can happen if the sampling frequency is $f_s = 150$ Hz, in which case the discrete-time signal is

$$\begin{aligned} x[n] = x(n/150) &= 2\cos(2\pi(50)(n/150) + \pi/2) + \cos(2\pi(150)(n/150)) \\ &= 2\cos(2\pi n/3 + \pi/2) + \cos(2\pi n) \\ &= 2\cos(2\pi n/3 + \pi/2) + 1 \end{aligned}$$

When $x[n]$ is reconstructed by the D/A converter running at $f_s = 150$ Hz, the final output will be

$$y(t) = x[n]\Big|_{n \to f_s t} = 2\cos(2\pi(150t)/3 + \pi/2) + 1 = 2\cos(2\pi(50)t + \pi/2) + 1$$

McClellan, Schafer and Yoder, *Signal Processing First*, ISBN 0-13-065562-7.
Pearson Prentice Hall, Inc. Upper Saddle River, NJ 07458. © 2003

PROBLEM 4.14:

(a) Assume that the disk is rotating $\boxed{\text{clockwise}}$ at a constant speed of 13 revolutions per second. If the flashing rate is 15 times per second, express the movement of the spot on the disk as a complex phasor, $p[n]$, that gives the position of the spot at the n-th flash. Assume that the spot is at the top when $n = 0$ (the first flash).

$$p(t) = r e^{j\pi/2} e^{-j2\pi(13)t}$$

INITIAL PHASE $= \pi/2$

$$p[n] = r e^{j\pi/2} e^{-j2\pi(\frac{13}{15})n}$$

$$= r e^{j\pi/2} e^{+j2\pi(\frac{2}{15})n}$$

ADD $2\pi n$ in EXPONENT.

(b) For the conditions in part (a), determine the apparent speed (in revolutions per second) and direction of movement of the "strobed" spot.

Convert $e^{j2\pi(\frac{2}{15})n}$ back to "continuous-time"

Replace n with $15t$

$\Rightarrow e^{j2\pi(2)t}$

$\therefore$ speed $= 2$ rev/sec

direction = COUNTER-CLOCKWISE
because sign of exponent is positive

(c) *Now assume that the rotation speed of the disk is unknown.* If the flashing rate is 13 times per second, and the spot on the disk moves $\boxed{\text{counter-clockwise}}$ by 15 degrees with each flash, determine the rotation speed of the disk (in rev/sec). If the answer is not unique give $\boxed{\text{all}}$ possible rotation speeds.

15° per flash $\Rightarrow$ 1 rev per 24 flashes

$\Rightarrow p[n] = e^{j2\pi n/24}$ (i.e., period = 24)

$\hat{p}(t) = e^{j2\pi f_0 t}$ where $f_0 =$ unknown speed.

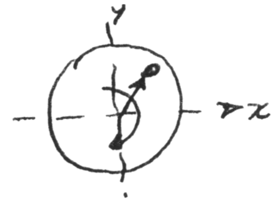
sample at 13 flashes/sec.

$$e^{j2\pi f_0 n/13} = e^{j2\pi n/24} \cdot e^{j2\pi \ell n}$$

$$\frac{f_0}{13} = \frac{1}{24} + \ell \Longrightarrow \boxed{f_0 = \frac{13}{24} + 13\ell} \quad \ell = \text{integer}$$

PROBLEM 4.15:

(a) 720 rpm = 12 rotations/sec.



If (x, y) = the co-ordinates of the spot we can also use polar co-ords: $r\angle\theta$.

The radius of the spot is constant: r

The angle of the vector from the origin to the spot changes LINEARLY

$$\theta = \varphi_0 + \omega_0 t$$

where φ_0 is the initial phase

ω_0 = freq of rotation in rad/sec.

$\therefore \quad \omega_0 = 2\pi(12) = 24\pi.$

So the position of the spot $= re^{-j24\pi t + j\varphi_0}$

The minus sign is for clockwise rotation

(b) Disk spot will stand still if flash rate is once per rotation, once every 2 rotations etc. $\Rightarrow$ possible flash rate to hold spot still $= \dfrac{12}{\ell}$, $\ell = 1, 2, 3, \ldots$

So we get an answer of $\boxed{\text{rate} = 1, 2, 3, 4, 6, \text{ \& } 12 \text{ per sec}}$

The rate must be a factor of 12

(c) $f_s = 13$ per sec. $\Rightarrow$ sample at $n/13$.

$x[n] = re^{j\varphi_0} e^{-j24\pi n/13} \quad \Rightarrow \omega_0 = \dfrac{-24\pi}{13}$

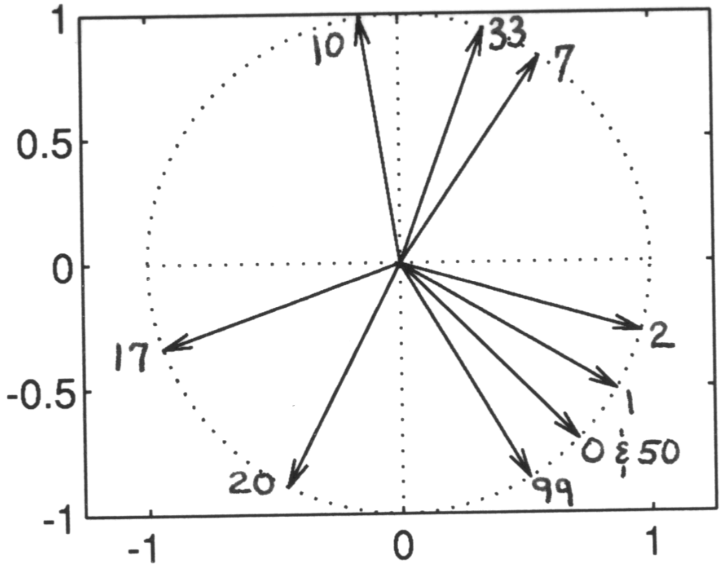
But ω_0 is same as: $-\dfrac{24\pi}{13} + 2\pi = 2\pi/13$

$x[n] = re^{j\varphi_0} e^{+j2\pi n/13} \longleftarrow$ (will take 13 flashes per revolution)

Spot will move counter-clockwise (due to + sign)

At what rate? once per sec = 60 rpm

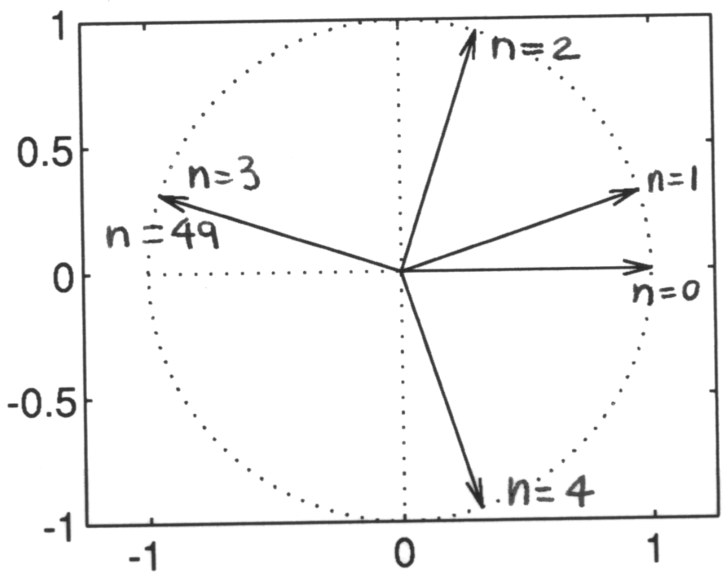
PROBLEM 4.16:



$$\text{Period} = \frac{2\pi}{0.08\pi} = 25$$

$$\Rightarrow z[50] = z[0]$$

$$z[33] = z[8]$$



When $n=7$

$$e[7] = e^{j0.1\pi(49)}$$

$$= e^{j4.9\pi}$$

$$= e^{j0.9\pi}$$

NOT PERIODIC

McClellan, Schafer and Yoder, *Signal Processing First*, ISBN 0-13-065562-7.
Pearson Prentice Hall, Inc. Upper Saddle River, NJ 07458. © 2003

PROBLEM 4.17:

(a) $\theta[n] = \pi(0.7\times 10^{-3})n^2$ ← This is $\theta[n]$ in $\Re e\{e^{j\theta[n]}\}$

For $n=10$:

$$\theta[10] = \pi(0.7\times 10^{-3})\,10^2 = 0.07\pi = 12.6^\circ$$

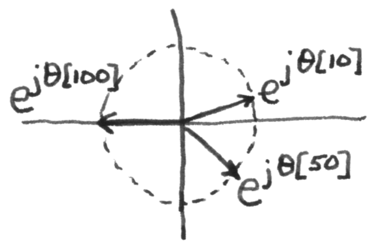
For $n=50$:

$$\theta[50] = \pi(0.7\times 10^{-3})(50)^2 = \pi(0.7\times 10^{-3}\times 25\times 10^2) = 1.75\pi$$

315°

For $n=100$:

$$\theta[100] = \pi(0.7\times 10^{-3})10^4 = 7\pi = \pi \text{ rads, or } 180^\circ$$



(c) Work part (c) before part (b)

$v[n] = \cos(0.7\pi n)$ $\quad f_s = 8000\,\text{Hz}$

Ideal D/A ⇒ replace n with $f_s t$

$$v(t) = v[n]\Big|_{n=8000t} = \cos(0.7\pi\times 8000t) = \cos(2\pi(2800)t)$$

Freq is 2800 Hz

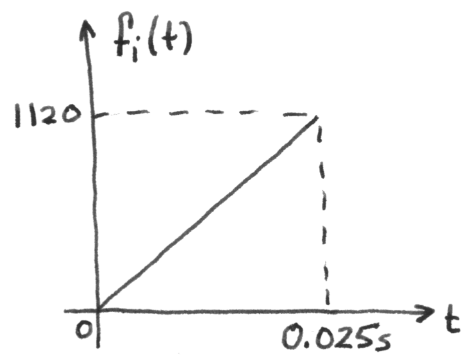
(b) $x[n] = \cos(\pi(0.7\times 10^{-3})n^2)$ ← Replace n with $8000t$

$$x(t) = \cos(\pi(0.7\times 10^{-3})\times 64\times 10^6\, t^2)$$
$$= \cos(\pi(44.8\times 10^3)t^2)$$

$$\underbrace{n = 0, 1, \ldots 200}$$
$$0 \le t \le \frac{200}{f_s} = 0.025\,\text{sec.}$$

$$\psi(t) = \pi(44.8\times 10^3)t^2$$

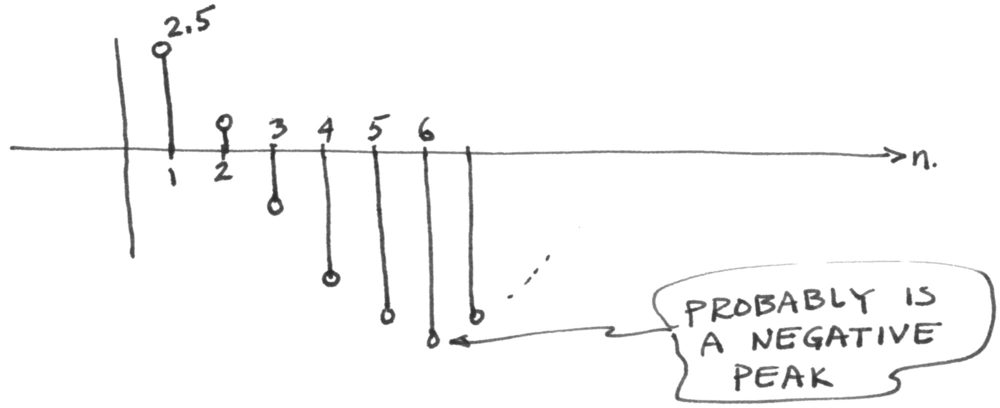
$$f_i(t) = \frac{1}{2\pi}\frac{d}{dt}\psi(t)$$
$$= \frac{1}{2\pi}(2\pi)(44.8\times 10^3)t$$
$$= 44800\,t \quad \text{Hz}$$



McClellan, Schafer and Yoder, *Signal Processing First*, ISBN 0-13-065562-7.
Pearson Prentice Hall, Inc. Upper Saddle River, NJ 07458. © 2003

PROBLEM 4.18:

(a)



(b) $\beta \cos n\omega_0 = \cos(n+1)\omega_0 + \cos(n-1)\omega_0$

$$\beta\left(\frac{e^{jn\omega_0}+e^{-jn\omega_0}}{2}\right) = \frac{e^{j(n+1)\omega_0}+e^{-j(n+1)\omega_0}}{2} + \frac{e^{j(n-1)\omega_0}+e^{-j(n-1)\omega_0}}{2}$$

$$\Rightarrow \beta\left(e^{jn\omega_0}+e^{-jn\omega_0}\right) = \left(e^{j\omega_0}+e^{-j\omega_0}\right)e^{jn\omega_0} + \left(e^{j\omega_0}+e^{-j\omega_0}\right)e^{-jn\omega_0}$$

$$\Rightarrow \beta = e^{j\omega_0}+e^{-j\omega_0} = 2\cos\omega_0$$

This result can be generalized to cosines with any phase so we get a method for extracting the frequency directly from $x[n]$ when we know that $x[n]$ is $x[n] = A\cos(\omega_0 n + \varphi)$.

The method is

$$\cos\omega_0 = \frac{x[n+1] + x[n-1]}{2x[n]}$$

where we use any 3 consecutive values in $x[n]$.

McClellan, Schafer and Yoder, *Signal Processing First*, ISBN 0-13-065562-7.
Pearson Prentice Hall, Inc. Upper Saddle River, NJ 07458.

$$\cos\omega_0 = \frac{x[5]+x[7]}{2x[6]} = \frac{-4.5677-4.5677}{2(-5.0)} = .9135$$

$$\Rightarrow \omega_0 = \cos^{-1}(0.9135) = 0.4190 = 2\pi/15$$

Now set up simultaneous eqns for A & φ

Let $Z = \frac{1}{2}Ae^{j\varphi}$

$$x[n] = Ze^{j\omega_0 n} + Z^* e^{-j\omega_0 n}$$

$$\begin{bmatrix} e^{j\omega_0} & e^{-j\omega_0} \\ e^{j2\omega_0} & e^{-j2\omega_0} \end{bmatrix} \begin{bmatrix} Z \\ Z^* \end{bmatrix} = \begin{bmatrix} x[1] \\ x[2] \end{bmatrix}.$$

Invert 2x2 matrix:

$$\begin{bmatrix} Z \\ Z^* \end{bmatrix} = \frac{1}{e^{-j\omega_0} - e^{j\omega_0}} \begin{bmatrix} e^{-j2\omega_0} & -e^{-j\omega_0} \\ -e^{j2\omega_0} & e^{j\omega_0} \end{bmatrix} \begin{bmatrix} 2.50 \\ .5226 \end{bmatrix}.$$

$$\therefore Z = \frac{e^{-j2\omega_0}x[1] - e^{-j\omega_0}x[2]}{e^{-j\omega_0} - e^{j\omega_0}} = \frac{1}{2}Ae^{j\varphi}$$

Now, plug in the numbers & CRUNCH:

$$\frac{1}{2}Ae^{j\varphi} = 2.023 + j1.47 = 2.5e^{j\pi/5}$$

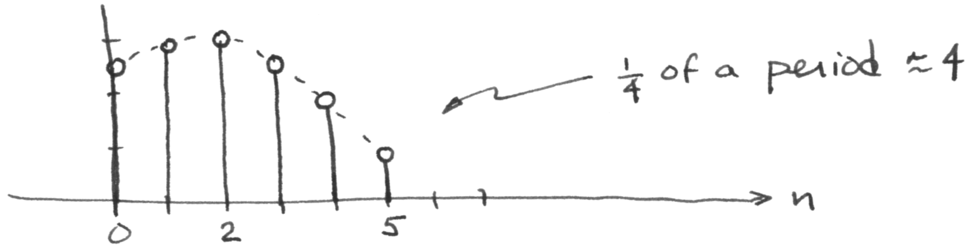
$$\Rightarrow A = 5$$

$$\varphi = \pi/5 \qquad\qquad 5\cos\left(\frac{2\pi}{15}n + \pi/5\right)$$

$$\omega_0 = 2\pi/15$$

PROBLEM 4.19:

You could estimate the values from a plot.



Looks like $A \approx 3$ $\quad \omega_0 \approx 2\pi\left(\frac{1}{\text{period}}\right) = 2\pi\frac{1}{16} = \frac{\pi}{8}$

$$\varphi = -2\pi\left(\frac{t_1}{T}\right) \cong -2\pi\left(\frac{2}{16}\right) = -\pi/4$$

EXACT:

Write 3 consecutive values of $x[n]$.

$$x[n-1] = \frac{A}{2}e^{j\varphi}e^{j\omega_0 n}e^{-j\omega_0} + \frac{A}{2}e^{-j\varphi}e^{-j\omega_0 n}e^{j\omega_0}$$

$$x[n] = \frac{A}{2}e^{j\varphi}e^{j\omega_0 n} + \frac{A}{2}e^{-j\varphi}e^{-j\omega_0 n}$$

$$x[n+1] = \frac{A}{2}e^{j\varphi}e^{j\omega_0 n}e^{j\omega_0} + \frac{A}{2}e^{-j\varphi}e^{-j\omega_0 n}e^{-j\omega_0}$$

$$\Rightarrow x[n-1] + x[n+1] = \frac{A}{2}e^{j\varphi}e^{j\omega_0 n}(2\cos\omega_0) + \frac{A}{2}e^{-j\varphi}e^{-j\omega_0 n}(2\cos\omega_0)$$

$$= (2\cos\omega_0)\, x[n].$$

$$\Rightarrow \cos\omega_0 = \frac{x[n-1]+x[n+1]}{2x[n]} = \frac{2.4271 + 2.9816}{2(2.9002)} = 0.9325$$

$$\Rightarrow \boxed{\omega_0 = 2\pi/17}$$

Let $z = \frac{A}{2}e^{j\varphi}$

$$x[0] = z + z^* = 2.4271$$

$$x[1] = e^{j2\pi/17}z + e^{-j2\pi/17}z^* = 2.9002$$

2 EQNS in 2 UNKNOWNS

$$z = 1.5e^{-j\pi/5}$$

$$\boxed{A = 3 \;\; \& \;\; \varphi = -\pi/5}$$

McClellan, Schafer and Yoder, *Signal Processing First*, ISBN 0-13-065562-7.
Pearson Prentice Hall, Inc. Upper Saddle River, NJ 07458. © 2003

PROBLEM 4.19 (more):

SOLVE SIMULTANEOUS EQNS FOR A & φ

Let $Z = \frac{1}{2} A e^{j\varphi}$

$$\begin{bmatrix} 1 & 1 \\ e^{j2\pi/17} & e^{-j2\pi/17} \end{bmatrix} \begin{bmatrix} Z \\ Z^* \end{bmatrix} = \begin{bmatrix} x[0] \\ x[1] \end{bmatrix}$$

Invert the 2x2 matrix: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

$$\Rightarrow \begin{bmatrix} Z \\ Z^* \end{bmatrix} = \frac{1}{e^{-j2\pi/17} - e^{j2\pi/17}} \begin{bmatrix} e^{-j2\pi/17} & -1 \\ -e^{+j2\pi/17} & 1 \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \end{bmatrix}$$

$$Z = \frac{x[0] e^{-j2\pi/17} - x[1]}{-2j \sin 2\pi/17} = \frac{1}{2} A e^{j\varphi}$$

PLUG IN $\left\{ \begin{matrix} x[0] = 2.4271 \\ x[1] = 2.9002 \end{matrix} \right\}$ AND SIMPLIFY

$$\Rightarrow Z = 1.2135 - j0.8817 = 1.5 e^{-j\pi/5}$$

$$\Rightarrow A = 3 \qquad \varphi = -\pi/5$$

McClellan, Schafer and Yoder, *Signal Processing First*, ISBN 0-13-065562-7.
Pearson Prentice Hall, Inc. Upper Saddle River, NJ 07458.