

3.4 Cubic Spline Interpolation

1) Spline approximation

So far we were finding one polynomial which fits all the points. A drawback of this approach is that this polynomial oscillates around the function and may deviate significantly from the function especially in the end intervals see fig. 3.12 on p. 151.

Alternative approach is to divide the interval into subintervals and use different approximating polynomials on each subinterval.

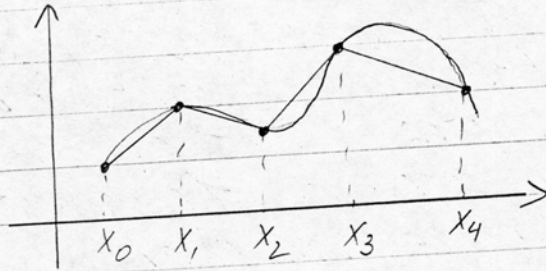
Def: Dividing the interval into subintervals and approximating by different (low degree) polynomials is called piecewise-polynomial approximation.

Ex: The simplest piecewise-polynomial approx. is the piecewise-linear interpolation.

Given the points

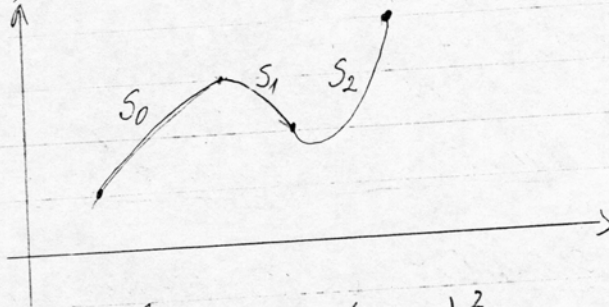
x_0	x_1	...	x_n
$f(x_0)$	$f(x_1)$...	$f(x_n)$

Each 2 of them are connected by a line.



Drawback: No differentiability at the endpoints of the subintervals. Thus, the interpolating function is not "smooth".

Ex: We can fit quadratic polynomials on $[x_i, x_{i+1}]$



$$S_i(x) = a_i + b_i(x-x_i) + c_i(x-x_i)^2$$

$$\text{Since } S_i(x_i) = a_i = f_i$$

$$S_i(x_{i+1}) = f_{i+1}$$

are only 2 conditions we can pose more conditions to ensure

differentiability. Say, we want

$$S_0'(x_1) = S_1'(x_1)$$

$$S_1'(x_2) = S_2'(x_2)$$

We have 9 parameters. The number of conditions is $6+2=8$. What could be the last condition? No enough parameters to prescribe the values of the derivatives.

2) Cubic spline approximations:

Def. The most common piecewise polynomial uses cubic polynomials in the interval $[x_i, x_{i+1}]$ and is called cubic spline interpolation.

There are different requirements that can be posed on the cubic spline interpolant. In all cases we expect the interpolant to be a differentiable function on the entire interval.

Let f be defined on $[a, b]$ and
 $a = x_0 < x_1 < \dots < x_n = b$

Let $S_j(x)$ be a cubic polynomial on $[x_j, x_{j+1}]$
 $j=0, \dots, n-1$

There are n such polynomials

$$S_j(x) = a_j + b_j(x-x_j) + c_j(x-x_j)^2 + d_j(x-x_j)^3$$

Thus, there are $4n$ parameters to determine
 $j=0, \dots, n-1$

2A) Suppose f is differentiable and we prescribe

$$\begin{cases} S_j(x_j) = f(x_j) = f_j \\ S_j(x_{j+1}) = f(x_{j+1}) = f_{j+1} \\ S_j'(x_j) = f'(x_j) = f_j' \\ S_j'(x_{j+1}) = f'(x_{j+1}) = f_{j+1}' \end{cases}$$

How many conditions? $4n$
Using these conditions we can uniquely determine the coefficients

a_j, b_j, c_j, d_j

namely, denote by $h_j = x_{j+1} - x_j$

$$a_j = f_j$$

$$b_j = f_j'$$

$$d_j = \frac{f_{j+1} h_j + f_j' h_j - 2(f_{j+1} - f_j)}{h_j^3}$$

$$= \frac{f_{j+1}' + f_j' - 2f[x_j, x_{j+1}]}{h_j^2}$$

$$c_j = \frac{f[x_j, x_{j+1}] - f_j'}{h_j} - h_j d_j$$

2B) Suppose we prescribe

$$(a) \quad \left\{ \begin{array}{l} S_j(x_j) = f(x_j) = f_j \quad j = 0, \dots, n \end{array} \right.$$

$$(b) \quad \left\{ \begin{array}{l} S_j(x_{j+1}) = S_{j+1}(x_{j+1}) = f_{j+1} \quad j = 0, \dots, n-2 \end{array} \right.$$

$$(c) \quad \left\{ \begin{array}{l} S_j'(x_{j+1}) = S_{j+1}'(x_{j+1}) \quad j = 0, \dots, n-2 \end{array} \right.$$

$$(d) \quad \left\{ \begin{array}{l} S_j''(x_{j+1}) = S_{j+1}''(x_{j+1}) \quad j = 0, \dots, n-2 \end{array} \right.$$

So we have $4n$ parameters. How many conditions

$$n+1 + 3(n-1) = n+1 + 3n-3 = 4n-2$$

we need 2 more conditions.

If we assign them as

$$S''(x_0) = S''(x_n) = 0$$

We say that we have free or natural boundary.

Def: Then the spline $S(x)$ is uniquely determined and is called natural spline.

2c) If we assign the same conditions as (*) plus the two conditions

$$S'(x_0) = f'(x_0)$$

$$S'(x_n) = f'(x_n)$$

We say that we have clamped boundary.

Def: The spline obtained with clamped boundary conditions is called complete spline interpolant (clamped cubic spline).

Note: Other ways of prescribing the 2 conditions also exist.

3) Computing with natural and complete splines.

Theorem 3.11. If f is defined at
 $a = x_0 < x_1 < \dots < x_n = b$
 then f has a unique natural spline
 interpolant $S(x)$ on the nodes
 x_0, x_1, \dots, x_n
 which satisfies $S''(a) = S''(b) = 0$.

Adding the 2 conditions to the system
 (•) we get

$$c_0 = c_n = 0$$

Theorem 3.12 If f is defined at
 $a = x_0 < x_1 < \dots < x_n = b$
 and differentiable at a and b
 then f has a unique clamped spline
 interpolant on the nodes
 x_0, x_1, \dots, x_n
 satisfying $S'(a) = f'(a)$ $S'(b) = f'(b)$.

To the system (•) we have to add
 the equations

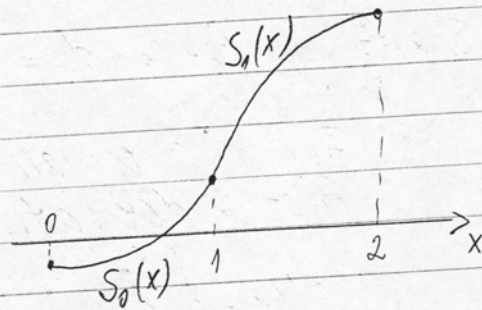
$$2h_0c_0 + h_0c_1 = \frac{3}{h_0}(f_1 - f_0) - 3f'(a)$$

$$h_{n-1}c_{n-1} + 2h_{n-1}c_n = 3f'(b) - \frac{3}{h_{n-1}}(f_n - f_{n-1})$$

Ex. Construct a natural (free) cubic spline $S(x)$ for the data

x	$f(x)$
0	$-\frac{1}{3}$
1	1
2	$\frac{11}{3}$

Solution:



We look for the spline in the form

$$S(x) = \begin{cases} S_0(x) = a_0 + b_0x + c_0x^2 + d_0x^3 & 0 \leq x < 1 \\ S_1(x) = a_1 + b_1(x-1) + c_1(x-1)^2 + d_1(x-1)^3 & 1 \leq x < 2 \end{cases}$$

We have 8 parameters: $a_0, b_0, c_0, d_0, a_1, b_1, c_1, d_1$.
We have the following conditions on $S_0(x), S_1(x)$

$$\begin{array}{cccc} S_0(0) = -\frac{1}{3} & S_1(1) = 1 & S_0'(1) = S_1'(1) & S_0''(0) = 0 \\ S_0(1) = 1 & S_1(2) = \frac{11}{3} & S_0''(1) = S_1''(1) & S_1''(2) = 0 \end{array}$$

So we have 8 conditions. These conditions give
free boundary

$$S_0(0) = -\frac{1}{3} \quad a_0 = -\frac{1}{3}$$

$$S_0(1) = 1 \quad -\frac{1}{3} + b_0 + c_0 + d_0 = 1 \Rightarrow b_0 + c_0 + d_0 = \frac{4}{3}$$

$$S_1(1) = 1 \quad a_1 = 1$$

$$S_1(2) = \frac{11}{3} \quad 1 + b_1 + c_1 + d_1 = \frac{11}{3} \Rightarrow b_1 + c_1 + d_1 = \frac{8}{3}$$

For the remaining 4 conditions we need the first 2 derivatives:

$$S'(x) = \begin{cases} S_0'(x) = b_0 + 2c_0x + 3d_0x^2 & 0 \leq x \leq 1 \\ S_1'(x) = b_1 + 2c_1(x-1) + 3d_1(x-1)^2 & 1 \leq x \leq 2 \end{cases}$$

$$S''(x) = \begin{cases} S_0''(x) = 2c_0 + 6d_0x & 0 \leq x \leq 1 \\ S_1''(x) = 2c_1 + 6d_1(x-1) & 1 \leq x \leq 2 \end{cases}$$

$$S_0'(1) = S_1'(1) \quad b_0 + 2c_0 + 3d_0 = b_1$$

$$S_0''(1) = S_1''(1) \quad 2c_0 + 6d_0 = 2c_1 \Rightarrow 3d_0 = c_1$$

$$S_0''(0) = 0 \quad 2c_0 = 0 \Rightarrow c_0 = 0$$

$$S_1''(2) = 0 \quad 2c_1 + 6d_1 = 0 \Rightarrow c_1 + 3d_1 = 0$$

Thus, we get the system for b_0, d_0, b_1, c_1, d_1

$$b_0 + d_0 = \frac{4}{3}$$

$$b_1 + c_1 + d_1 = \frac{8}{3}$$

$$b_0 + 3d_0 = \frac{2}{3}b_1$$

$$3d_0 = c_1$$

$$c_1 + 3d_1 = 0$$

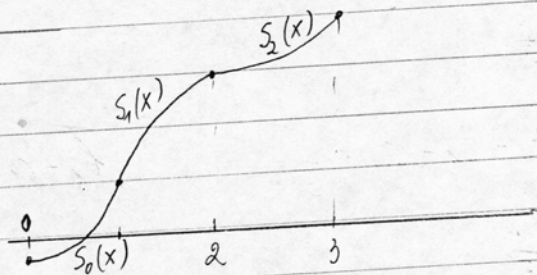
Solving this system we obtain

$$a_0 = -\frac{1}{3}, b_0 = 1, c_0 = 0, d_0 = \frac{1}{3}, a_1 = 1, b_1 = 2, c_1 = 1, d_1 = -\frac{1}{3}$$

Ex. Construct a clamped cubic spline $S(x)$ for the data

x	$f(x)$	$f'(x)$
0	$-\frac{1}{3}$	1
1	$\frac{3}{2}$	-
2	$\frac{19}{3}$	-
3	$\frac{33}{2}$	14

Solution:



We look for the spline in the form

$$S(x) = \begin{cases} S_0(x) = a_0 + b_0x + c_0x^2 + d_0x^3 & 0 \leq x \leq 1 \\ S_1(x) = a_1 + b_1(x-1) + c_1(x-1)^2 + d_1(x-1)^3 & 1 \leq x \leq 2 \\ S_2(x) = a_2 + b_2(x-2) + c_2(x-2)^2 + d_2(x-2)^3 & 2 \leq x \leq 3 \end{cases}$$

We have 12 parameters. We have the following conditions on $S_0(x)$, $S_1(x)$, $S_2(x)$

$$\begin{array}{lll} S_0(0) = -\frac{1}{3} & S_1(1) = \frac{3}{2} & S_2(2) = \frac{19}{3} = S_0'(0) = 1 \\ S_0(1) = \frac{3}{2} & S_1(2) = \frac{19}{3} & S_2(3) = \frac{33}{2} = S_2'(3) = 14 \end{array}$$

$$S_0'(1) = S_1'(1)$$

$$S_0''(1) = S_1''(1)$$

$$S_1'(2) = S_2'(2)$$

$$S_1''(2) = S_2''(2)$$

So we have 12 conditions. These conditions give:

$$S_0(0) = -\frac{1}{3} : a_0 = -\frac{1}{3}$$

$$S_0(1) = \frac{3}{2} : -\frac{1}{3} + b_0 + c_0 + d_0 = \frac{3}{2} \Rightarrow b_0 + c_0 + d_0 = \frac{11}{6}$$

$$S_1(1) = \frac{3}{2} : a_1 = \frac{3}{2}$$

$$S_1(2) = \frac{19}{3} : \frac{3}{2} + b_1 + c_1 + d_1 = \frac{19}{3} \quad b_1 + c_1 + d_1 = \frac{29}{6}$$

$$S_2(2) = \frac{19}{3} : a_2 = \frac{19}{3}$$

$$S_2(3) = \frac{33}{2} : \frac{19}{3} + b_2 + c_2 + d_2 = \frac{33}{2} \quad b_2 + c_2 + d_2 = \frac{61}{6}$$

For the remaining 4 conditions we have to use the first 2 derivatives of the spline

$$S'(x) = \begin{cases} S_0'(x) = b_0 + 2c_0x + 3d_0x^2 & 0 \leq x \leq 1 \\ S_1'(x) = b_1 + 2c_1(x-1) + 3d_1(x-1)^2 & 1 \leq x \leq 2 \\ S_2'(x) = b_2 + 2c_2(x-2) + 3d_2(x-2)^2 & 2 \leq x \leq 3 \end{cases}$$

$$S_0'(0) = 1 : b_0 = 1$$

$$S_2'(3) = 14 : b_2 + 2c_2 + 3d_2 = 14$$

$$S''(x) = \begin{cases} S_0''(x) = 2c_0 + 6d_0x & 0 \leq x < 1 \\ S_1''(x) = 2c_1 + 6d_1(x-1) & 1 \leq x < 2 \\ S_2''(x) = 2c_2 + 6d_2(x-2) & 2 \leq x < 3 \end{cases}$$

$$S_0'(1) = S_1'(1): 1 + 2c_0 + 3d_0 = b_1$$

$$S_1'(2) = S_2'(2): b_1 + 2c_1 + 3d_1 = b_2$$

$$S_0''(1) = S_1''(1): 2c_0 + 6d_0 = 2c_1$$

$$S_1''(2) = S_2''(2): 2c_1 + 6d_1 = 2c_2$$

Thus, we get the following system for the coefficients

$$\begin{cases} 1 + c_0 + d_0 = \frac{11}{6} \\ b_1 + c_1 + d_1 = \frac{29}{6} \\ b_2 + c_2 + d_2 = \frac{61}{6} \\ b_2 + 2c_2 + 3d_2 = 14 \\ 1 + 2c_0 + 3d_0 = b_1 \\ b_1 + 2c_1 + 3d_1 = b_2 \\ 2c_0 + 6d_0 = 2c_1 \\ 2c_1 + 6d_1 = 2c_2 \end{cases}$$

8 equations for $c_0, d_0, b_1, c_1, d_1, b_2, c_2, d_2$.
Solving this linear system we get

$$b_0 = 1 \quad c_0 = \frac{1}{2} \quad d_0 = \frac{1}{3}$$

$$b_1 = 3 \quad c_1 = \frac{3}{2} \quad d_1 = \frac{1}{3}$$

$$b_2 = 7 \quad c_2 = \frac{5}{2} \quad d_2 = \frac{2}{3}$$

Thus, finally the spline is

$$S(x) = \begin{cases} S_0(x) = -\frac{1}{3} + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 & 0 \leq x \leq 1 \\ S_1(x) = \frac{3}{2} + 3(x-1) + \frac{3}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 & 1 \leq x \leq 2 \\ S_2(x) = \frac{19}{3} + 7(x-2) + \frac{5}{2}(x-2)^2 + \frac{2}{3}(x-2)^3 & 2 \leq x \leq 3 \end{cases}$$

Ex #7/153 A natural cubic spline S on $[0, 2]$ is defined by

$$S(x) = \begin{cases} S_0(x) = 1 + 2x - x^3 & 0 \leq x \leq 1 \\ S_1(x) = 2 + b(x-1) + c(x-1)^2 + d(x-1)^3 & 1 \leq x \leq 2 \end{cases}$$

Find b, c, d .

From the condition $S_0'(1) = S_1'(1)$ we have

$$\begin{aligned} S_0'(x) &= 2 - 3x^2 & S_0'(1) &= -1 \\ S_1'(x) &= b + 2c(x-1) + 3d(x-1)^2 & S_1'(1) &= b \end{aligned} \Rightarrow \boxed{b = -1}$$

From the condition that $S_0''(1) = S_1''(1)$

$$\begin{aligned} S_0''(x) &= -6x & S_0''(1) &= -6 \\ S_1''(x) &= 2c + 6d(x-1) & S_1''(1) &= 2c \end{aligned} \Rightarrow \boxed{c = -3}$$

From the condition $S_1''(2) = 0$ we have

$$S_1''(2) = -6 + 6d = 0 \Rightarrow \boxed{d = 1}$$