Chapter 6 Direct Methods for Solving Linear systems

6.1 Linear System of Equations

The point of this chapter is solving linear systems:

$$
\begin{array}{ll}
E_1 & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_2 = b_1 \\
E_2 & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_2 = b_2 \\
\vdots & \cdots \\
E_n & a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n
\end{array}
$$

where the constants a_{ij}, $i=1,\dots,n$, $j=1,\dots,n$ and b_i $i=1,\dots,n$ are given and $x_1,\dots,x_n$ are unknown

Ex.

$$
\begin{aligned}
x_1 - x_2 + 3x_3 &= 2 \\
x_1 + x_2 \phantom{{}+3x_3} &= 3 \\
3x_1 - 3x_2 + x_3 &= -1
\end{aligned}
$$

<u>Def:</u> <u>Direct methods</u> are methods that give an answer in a fixed number of steps, subject only to round off error.

A linear system might have no solution, one solution or many solutions.

The following 3 operations transform a system into system that has the same solutions (but might be easier to solve).

I Equation E_i can be multiplied by a nonzero constant λ with the resulting equation used in place of E_i. This operation is denoted $(\lambda E_i) \rightarrow (E_i)$

Ex:
$$\begin{aligned} x_1 - x_2 + 3x_3 &= 2 \\ x_1 + x_2 \qquad &= 3 \qquad (\tfrac{1}{3} E_3) \rightarrow (E_3) \\ x_1 - x_2 + \tfrac{1}{3} x_3 &= -\tfrac{1}{3} \end{aligned}$$

II Equation E_j can be multiplied by any constant λ and added to equation E_i with the resulting equation used in place of E_i. This operation is denoted by

$$(E_i + \lambda E_j) \rightarrow (E_i)$$

Ex:
$$\begin{aligned} x_1 - x_2 + 3x_3 &= 2 \\ x_1 + x_2 \qquad &= 3 \qquad (E_3 + (-3)E_1) \rightarrow (E_3) \\ -8x_3 &= -7 \end{aligned}$$

III Equations E_i and E_j can be interchanged in order. This operation is denoted

$$(E_i) \leftrightarrow (E_j)$$

Ex:

$$\begin{aligned} x_1 - x_2 + 3x_3 &= 2 \\ 3x_1 - 3x_2 + x_3 &= -1 \\ x_1 + x_2 &= 3 \end{aligned} \qquad (E_2) \leftrightarrow (E_3)$$

The aim is by a sequence of operation to reduce the original system to a <u>triangular</u> (<u>reduced</u>) <u>form</u>:

$$\begin{aligned} \tilde{a}_{11} x_1 + \tilde{a}_{12} x_2 + \dots + \tilde{a}_{1n} x_n &= \tilde{b}_1 \\ \tilde{a}_{22} x_2 + \dots + \tilde{a}_{2n} x_n &= \tilde{b}_2 \\ - - - - - - - - - - \\ \tilde{a}_{nn} x_n &= \tilde{b}_n \end{aligned}$$

This system has the same solutions as the original one and can be solved easily by a procedure called <u>backward-substitution process</u>.

Ex:

$$\begin{aligned} x_1 - x_2 + 3x_3 &= 2 \\ x_1 + x_2 &= 3 \\ 3x_1 - 3x_2 + x_3 &= -1 \end{aligned}$$

$$(E_3 + (-3)E_1) \rightarrow E_3$$

$$x_1 - x_2 + 3x_3 = 2$$
$$2x_2 - 3x_3 = -1 \qquad (E_2 + (-1)E_1) \to (E_2)$$
$$-8x_3 = -7$$

This is in <u>triangular form</u>. Now we use backward substitution to solve.

$$x_3 = \frac{7}{8} \qquad 2x_2 = 1 + 3 \cdot \frac{7}{8} = 1 + \frac{21}{8} = \frac{29}{8}$$
$$x_2 = \frac{29}{16}$$

To simplify the computation we add the second equation to the first

$$x_1 + x_2 \quad = 3$$
$$x_1 = 3 - x_2 = 3 - \frac{29}{16} = \frac{48-29}{16} = \frac{19}{16}$$

Thus the solution is

$$x_1 = \frac{19}{16} \qquad x_2 = \frac{29}{16} \qquad x_3 = \frac{7}{8}$$

When performing the operations we do not want to write out the full equations or to carry the variables x_1, x_2, x_3.

Def: An $n \times m$ matrix is a rectangular array of elements with n rows and m column in which not only is the value of an element important, but also its position in the array.

Matrices are usually denoted with capital letters: A, B, C; their elements with small letters

$$a_{ij} \quad i^{th} \text{ row} \quad j^{th} \text{ column}$$

Thus,

$$A = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ - & - & - & - \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}$$

Ex: The matrix

$$A = \begin{pmatrix} 1 & -2 & 3 & 5 \\ -1 & 0 & -7 & 4 \\ 2 & -3 & 9 & 8 \end{pmatrix}$$

is an 3×4 matrix (3 row, 4 columns).

$$a_{32} = -3 \qquad a_{14} = 5 \qquad a_{24} = 4$$

Def: The $1 \times n$ matrix

$$A = (a_1 \ a_2 \ \dots \ a_n)$$

is called n-dimensional row vector.

The $n \times 1$ matrix

$$A = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

is called an n-dimensional column vector.

Vectors are usually denoted by boldface lowercase letters or by

$$\vec{a} = (a_1, \dots, a_n)$$

If we set $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ then the system can be written in concise form

$$A\vec{x} = b$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ - & - & - & - \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

Def: From A and b we compose an $n \times (n+1)$ matrix

$$(A, b) = \left(\begin{array}{ccc|c} a_{11} & \dots & a_{1n} & b_1 \\ - & - & - & - \\ a_{n1} & \dots & a_{nn} & b_n \end{array}\right)$$

is called the augmented matrix.

Instead of performing the operations on the system, we perform them on the augmented matrix.

Ex: Use the augmented matrix to solve the system

$$x_1 - x_2 + 3x_3 = 2$$
$$x_1 + x_2 = 3$$
$$3x_1 - 3x_2 + x_3 = -1$$

We write the system in augmented form

$$\left(\begin{array}{ccc|c} 1 & -1 & 3 & 2 \\ 1 & 1 & 0 & 3 \\ 3 & -3 & 1 & -1 \end{array}\right) \xrightarrow[(E_2-3E_1)]{} \left(\begin{array}{ccc|c} 1 & -1 & 3 & 2 \\ 0 & 2 & -3 & 1 \\ 0 & 0 & -8 & -7 \end{array}\right)$$

Then we can use backward substitution to solve. This procedure for solving linear systems is common.

Def: The procedure involved in this process is called Gaussian elimination with backward substitution.

In general, it consists in the following

1) Provided $a_{11} \neq 0$ we use it to eliminate all elements under it.

$$\left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} & b_n \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ 0 & \cdots & \tilde{a}_{2n} & \tilde{b}_2 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & \tilde{a}_{nn} & \tilde{b}_n \end{array}\right)$$

This is done with the operations

$$\left(E_j - \frac{a_{j1}}{a_{11}} E_1\right) \longrightarrow (E_j)$$

$$j = 2, \ldots, n$$

Note: the entries in the rows $E_2, \ldots, E_n$ change but we denote them again by a_{ij}.

2) We continue the procedure using a_{ii} (provided $a_{ii} \neq 0$) to eliminate (make zeroes) all entries underneath

$$\left(E_j - \frac{a_{ji}}{a_{ii}} E_i\right) \longrightarrow (E_j) \quad j = i+1, \ldots, n.$$

3) This way we get a triangular system

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ 0 & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{nn} & b_n \end{array}\right)$$

Then we use backward substitution

$$x_n = \frac{b_n}{a_{nn}}$$

$$x_{n-1} = \frac{b_{n-1} - a_{n-1,n}x_n}{a_{n-1,n-1}}$$

$$- \quad - \quad - \quad - \quad -$$

$$x_i = \frac{b_i - a_{in}x_n - \cdots - a_{i,i+1}x_{i+1}}{a_{ii}}$$

for each $i = n-1, \cdots, 1$.

Def: The element a_{ii} in each step used for elimination of the entries beneath is called pivot element.

The procedure above requires that the pivot element is nonzero.

What happens if a pivot element is zero?

Ex: Consider the system

$$\begin{aligned} x_1 + x_2 \quad\quad\; + x_4 &= 2 \\ 2x_1 + x_2 - x_3 + x_4 &= 1 \\ -x_1 + 2x_2 + 3x_3 - x_4 &= 4 \\ 3x_1 - x_2 - x_3 + 2x_4 &= -3 \end{aligned}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 2 \\ 2 & 1 & -1 & 1 & 1 \\ -1 & 2 & 3 & -1 & 4 \\ 3 & -1 & -1 & 2 & -3 \end{array}\right) \longrightarrow$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 2 \\ 0 & +1 & +1 & +1 & +3 \\ 0 & 3 & 3 & 0 & 6 \\ 0 & -4 & -1 & -1 & -9 \end{array}\right) \longrightarrow$$

pivot is zero

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & -3 & -3 \\ 0 & 0 & 3 & 3 & 3 \end{array}\right) \longrightarrow$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

By backward substitution we have

$x_4 = 1 \qquad x_3 + x_4 = 1 \Rightarrow x_3 = 0$

$x_2 + x_3 + x_4 = 3 \Rightarrow x_2 = 2$

$x_1 + x_2 + x_4 = 2 \Rightarrow x_1 = -1$

Thus, the solution is

$x_1 = -1 \quad x_2 = 2, \quad x_3 = 0 \quad x_4 = 1.$

So, if a pivot happens to be zero we can search for a non-zero element in the column below and interchange the 2 rows.

What if there is no non-zero element below?

Ex. The purpose of this example is to show what else can happen and what conclusions can be derived from that. We consider the systems

$$\begin{aligned} x_1 - x_2 + x_3 &= 4 \\ 3x_1 - 3x_2 + x_3 &= 2 \\ -x_1 + x_2 - 3x_3 &= 6 \end{aligned} \qquad \begin{aligned} x_1 - x_2 + x_3 &= 4 \\ 3x_1 - 3x_2 + x_3 &= 2 \\ -x_1 + x_2 - 3x_3 &= -14 \end{aligned}$$

We perform computations simultaneously

$$\left(\begin{array}{ccc|c|c} 1 & -1 & 1 & 4 & 4 \\ 3 & -3 & 1 & 2 & 2 \\ -1 & 1 & -3 & 6 & -14 \end{array}\right) \longrightarrow$$

pivot is zero $\longrightarrow$ No other pivot

$$\left(\begin{array}{ccc|c|c} 1 & -1 & 1 & 4 & 4 \\ 0 & 0 & -2 & -10 & -10 \\ 0 & 0 & -2 & 10 & -10 \end{array}\right) \longrightarrow$$

$$\left(\begin{array}{ccc|c|c} 1 & -1 & 1 & 4 & 4 \\ 0 & 0 & 1 & 5 & 5 \\ 0 & 0 & 1 & -5 & 5 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c|c} 1 & -1 & 1 & 4 & 4 \\ 0 & 0 & 1 & 5 & 5 \\ 0 & 0 & 0 & -10 & 0 \end{array}\right)$$

Ist case $x_3 = -5$
$x_3 = 5$
inconsistent

IInd case $x_3 = 5$
$x_3 = 5$
$x_3 = 5$ $\quad x_1 - x_2 + 5 = 4$
$x_1 - x_2 = -1$
many solutions

Thus, if a pivot is zero and there is no other nonzero pivot the system either has no solution or has many solutions. Thus, we have a pathological case.

Ex. Given the linear system

$$2x_1 - 6\alpha x_2 = 3$$
$$3\alpha x_1 - x_2 = \frac{3}{2}$$

(a) Find the value(s) of α for which the system has no solution

(b) Find the value(s) of α for which the system has infinite number of solutions.

(c) Assuming a unique solution exists for a given α, find that solution

$$\begin{array}{c} \\ \left(\begin{array}{cc|c} \alpha & -6\alpha & 3 \\ 3\alpha & -1 & \frac{3}{2} \end{array}\right) \end{array} \to \begin{array}{ccc} -3\alpha & 9\alpha^2 & \frac{3}{2}(1-3\alpha) \end{array} \atop \left(\begin{array}{cc|c} \alpha & -6\alpha & 3 \\ 0 & 9\alpha^2-1 & \frac{3}{2}(1-3\alpha) \end{array}\right)$$

(a) The system will have no solutions if

$$9\alpha^2 - 1 = 0 \Rightarrow \alpha = \pm\frac{1}{3}$$

and

$$1-3\alpha \neq 0 \Rightarrow \alpha \neq \frac{1}{3}$$

Thus, the system has no solution if $\alpha = -\frac{1}{3}$

(b) The system has infinitely many solutions if

$$9\alpha^2-1=0 \Rightarrow \alpha = \pm\frac{1}{3}$$

and

$$1-3\alpha = 0 \Rightarrow \alpha = \frac{1}{3}$$

Thus, the system has infinitely many solutions for $\alpha = \frac{1}{3}$

(c) For all other values of α the system has a unique solution

$$x_2 = \frac{\frac{3}{2}(1-3\alpha)}{9\alpha^2-1} = \frac{\frac{3}{2}(1-3\alpha)}{(3\alpha-1)(3\alpha+1)} = -\frac{3}{2(3\alpha+1)}$$

From the first equation we have

$$2x_1 + (6\alpha)\frac{3}{\alpha(3\alpha+1)} = 3$$

$$\alpha x_1 = 3 - \frac{9\alpha}{3\alpha+1} = \frac{9\alpha+3-9\alpha}{3\alpha+1} = \frac{3}{3\alpha+1}$$

$$x_1 = \frac{3}{\alpha(3\alpha+1)}$$

The computational complexity of the numerical methods for solving linear systems is measured in number of operations:

Number of
- additions/subtractions
- multiplications/divisions

How many operations are required for the Gaussian elimination with backward substitutions

Let's count them first on a specific example.

Ex. Perform Gaussian elimination with backward substitution on the system and count the number of operations performed.

$$4x_1 + x_2 + 2x_3 = 9$$
$$2x_1 + 4x_2 - x_3 = -5$$
$$x_1 + x_2 - 3x_3 = -9$$

Consider the augmented matrix

$$\left(\begin{array}{ccc|c} 4 & 1 & 2 & 9 \\ 2 & 4 & -1 & -5 \\ 1 & 1 & -3 & -9 \end{array}\right) \xrightarrow{MD\ 3+1} \qquad m_{21} = \frac{-2}{4} = -\frac{1}{2}$$

$$\left(\begin{array}{ccc|c} -2 & -\frac{1}{2} & -1 & -\frac{9}{2} \\ 2 & 4 & -1 & -5 \\ 1 & 1 & -3 & -9 \end{array}\right) \xrightarrow{AS\ 3}$$

$$\left(\begin{array}{ccc|c} 4 & 1 & 2 & 9 \\ 0 & \frac{7}{2} & -2 & -\frac{19}{2} \\ 1 & 1 & -3 & -9 \end{array}\right) \xrightarrow{MD\ 3+1} \qquad m_{31} = -\frac{1}{4}$$

$$\left(\begin{array}{ccc|c} -1 & -\frac{1}{4} & -\frac{1}{2} & -\frac{9}{4} \\ 0 & \frac{7}{2} & -2 & -\frac{19}{2} \\ 1 & 1 & -3 & -9 \end{array}\right) \xrightarrow{AS\ 3}$$

$$\left(\begin{array}{ccc|c} 4 & 1 & 2 & 9 \\ 0 & \frac{7}{2} & -2 & -\frac{19}{2} \\ 0 & \frac{3}{4} & -\frac{7}{2} & -\frac{45}{4} \end{array}\right) \xrightarrow{MD\ 2+1} \qquad m_{32} = \frac{-\frac{3}{4}}{\frac{7}{2}} = -\frac{3}{14}$$

$$\left(\begin{array}{ccc|c} 4 & 1 & 2 & 9 \\ 0 & -\frac{3}{4} & \frac{12}{28} & \frac{57}{28} \\ 0 & \frac{3}{4} & -\frac{7}{2} & -\frac{45}{4} \end{array}\right) \xrightarrow{AS\ 2}$$

$$\left(\begin{array}{ccc|c} 4 & 1 & 2 & 9 \\ 0 & \frac{7}{2} & -2 & -\frac{19}{2} \\ 0 & 0 & -\frac{86}{28} & -\frac{258}{28} \end{array}\right)$$

Backward substitution

$$X_3 = \frac{-\frac{258}{28}}{-\frac{86}{28}} = 3 \qquad MD\ 1$$

$$\left.\begin{array}{ll} \frac{7}{2} X_2 - 2 \cdot 3 = -\frac{19}{2} & MD\ 1 \\ \frac{7}{2} X_2 = -\frac{19}{2} + 6 = -\frac{7}{2} & AS\ 1 \\ X_2 = -1 & MD\ 1 \end{array}\right\} \begin{array}{l} \text{total} \\ MD\ 2\ \ AS\ 1 \end{array}$$

$$4X_1 + X_2 + 2X_3 = 9$$

$$\begin{array}{ll} 4X_1 \cdot -1 + 2 \cdot 3 = 9 & MD\ 2 \\ 4X_1 = 9+1-6 = 4 & AS\ 2 \\ X_1 = 1 & MD\ 1 \end{array} \left\} \begin{array}{l} \text{total} \\ MD\ 3\ AS\ 2 \end{array}\right.$$

In general, we have the i^{th} equation

$$a_{ii} \quad a_{i\,i+1} \; - - - \; a_{in} \;|\; b_i$$

We want to eliminate with the pivot a_{ii} the element a_{ji} $(j>i)$. Thus, we have to compute the multiplyer of all elemen the i^{th} row: $m_{ji} = \frac{-a_{ji}}{a_{ii}}$

MD: 1

and then multiply all elements in i^{th} ro by m_{ji}: MD: $n+1-i$

Finally we add the i^{th} row to the j^{th} row

AS: $n+1-i$

Thus, to eliminate a_{ji} with pivot a_{ii} we need

MD $(n+2-i)$

AS $n+1-i$

All elements a_{ji} which we have to eliminate with pivot a_{ii} are $(n-i)$ in number. Thus, to eliminate all elements below a_{ii} we need

MD $(n+2-i)(n-i)$

AS $(n+1-i)(n-i)$

We have to do that for pivots a_{ii} for $i=1,2,..,n-1$.

Thus, for addition/subtr. we have

$$\sum_{i=1}^{n-1}(n+1-i)(n-i) = \sum_{i=1}^{n-1}(n-i)^2 + \sum_{i=1}^{n-1}(n-i)$$

$$= 1^2+2^2+\dots+(n-1)^2 + 1+2+\dots+(n-1)$$

$$= \frac{(n-1)(n)(2n-1)}{6} + \frac{(n-1)n}{2} =$$

$$= \frac{(2n^2-3n+1)n}{6} + \frac{n^2-n}{2} =$$

$$= \frac{2n^3-3n^2+n+3n^2-3n}{6} =$$

$$= \frac{2n^3-2n}{6} = \frac{n^3-n}{3}$$

Thus, for all eliminations we need

MD $\dfrac{2n^3+3n^2-5n}{6}$

AS $\dfrac{n^3-n}{3}$

For the backward substitution we use the formula

$$x_i = \frac{b_i - a_{in}x_n - \dots - a_{i,i+1}x_{i+1}}{a_{ii}}$$

Thus, we have

MD $n-i+1$

AS $n-i$

For computing all x_i we have

$$\sum_{i=1}^{n-1} n-i = 1+2+\dots+n-1 = \frac{n(n-1)}{2}$$

The number of

AS : $\frac{n(n-1)}{2}$

We have one more multiplication for the computation of each x_i, $i=1,\dots,n$

Thus the number of

$$MD: \quad \frac{n(n-1)}{2} + n = \frac{n^2-n+2n}{2} = \frac{n^2+n}{2}$$