PROBLEM 10.1:

$$y(t) = x(t - \tfrac{1}{2})$$

Let $x(t) = e^{j\omega t}$,

and determine the output

$$y(t) = e^{j\omega(t-1/2)}$$

$$= e^{-j\omega/2} e^{j\omega t} = H(j\omega) e^{j\omega t}$$

$$\therefore H(j\omega) = e^{-j\omega/2}$$

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PROBLEM 10.2:

$$H(j\omega) = \frac{3-j\omega}{3+j\omega} e^{-j\omega}$$

(a) $$|H(j\omega)|^2 = H(j\omega)H^*(j\omega) = \frac{3-j\omega}{3+j\omega} e^{-j\omega} \frac{3+j\omega}{3-j\omega} e^{j\omega}$$

$$\Rightarrow |H(j\omega)|^2 = 1 \quad \text{for all } \omega$$

(b) $$\angle H(j\omega) = \angle\text{Numerator} - \angle\text{Denominator} - \omega$$

$$= \text{Tan}^{-1}\left(\frac{-\omega}{3}\right) - \text{Tan}^{-1}\left(\frac{\omega}{3}\right) - \omega$$

from $e^{-j\omega}$

(c) $x(t) = 4 + \cos(3t)$

There are two freqs in $x(t)$: 0 and 3 rad/s

$\Rightarrow$ Evaluate $H(j\omega)$ at $\omega = 0$ and $\omega = 3$

$$H(j0) = \frac{3-j0}{3+j0} e^{-j0} = 1$$

$H(j3)$ has a magnitude of 1 (from part (a))

$$\angle H(j3) = \text{Tan}^{-1}\left(\frac{-3}{3}\right) - \text{Tan}^{-1}\left(\frac{3}{3}\right) - 3 \quad \text{(from part (b))}$$

$$= -\pi/4 - (\pi/4) - 3$$

$$= -\pi/2 - 3 \approx -4.571$$

If we add 2π, the phase becomes $\angle H(j3) = 1.712$

$$y(t) = 4 \cdot H(j0) + |H(j3)| \cos(3t + \angle H(j3))$$

$$= 4 + \cos(3t + 1.712)$$

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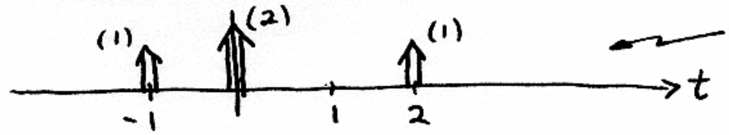
PROBLEM 10.3:

(a) Let $x(t) = \delta(t)$

Then $y(t) = \delta(t+1) + 2\delta(t) + \delta(t-2)$ ← This is $h(t)$



← Plot of $h(t)$

(b) $$H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} \{\delta(t+1) + 2\delta(t) + \delta(t-2)\} e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} \delta(t+1) e^{-j\omega t} dt + \int_{-\infty}^{\infty} 2\delta(t) e^{-j\omega t} dt + \int_{-\infty}^{\infty} \delta(t-2) e^{-j\omega t} dt$$

$$H(j\omega) = e^{-j\omega(-1)} + 2e^{-j\omega(0)} + e^{-j\omega(2)}$$

$$= e^{j\omega} + 2 + e^{-j2\omega}$$

(c) Let $x(t) = e^{j\omega t}$

Then $$y(t) = e^{j\omega(t+1)} + 2e^{j\omega t} + e^{j\omega(t-2)}$$

$$= e^{j\omega t} e^{j\omega} + 2e^{j\omega t} + e^{j\omega t} e^{-j2\omega}$$

$$= \underbrace{\{e^{j\omega} + 2 + e^{-j2\omega}\}}_{\text{This is } H(j\omega) \text{ which is the \underline{same} as in part (b).}} e^{j\omega t}$$

PROBLEM 10.4:

(a) $$H(j\omega) = \int_{-\infty}^{\infty} \{\delta(t) - 0.1e^{-0.1t}u(t)\} e^{-j\omega t}\,dt$$

$$= \underbrace{\int_{-\infty}^{\infty} \delta(t) e^{-j\omega t}\,dt}_{= e^{-j\omega(0)} = 1} - 0.1 \underbrace{\int_{-\infty}^{\infty} e^{-0.1t} u(t) e^{-j\omega t}\,dt}$$

$$\int_{0}^{\infty} e^{-0.1t} e^{-j\omega t}\,dt = \frac{e^{-(0.1+j\omega)t}}{-(0.1+j\omega)}\Bigg|_0^{\infty} = 0 - \frac{1}{-(0.1+j\omega)} = \frac{1}{0.1+j\omega}$$

Thus, $$H(j\omega) = 1 - \frac{0.1}{0.1+j\omega} = \frac{j\omega}{0.1+j\omega}$$

(b) $$|H(j\omega)|^2 = \left(\frac{j\omega}{0.1+j\omega}\right)\left(\frac{-j\omega}{0.1-j\omega}\right) = \frac{\omega^2}{0.01 + j0.1\omega - j0.1\omega - (j\omega)^2}$$

$$= \frac{\omega^2}{0.01+\omega^2}$$

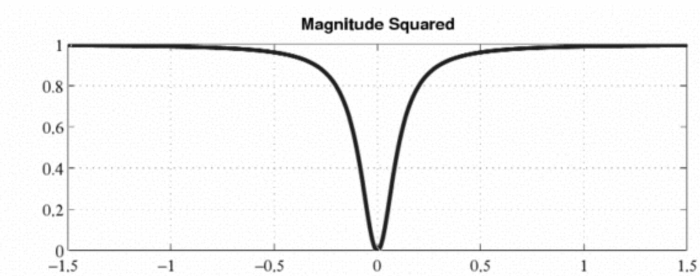
At $\omega = 0$, $|H(j0)|^2 = 0$

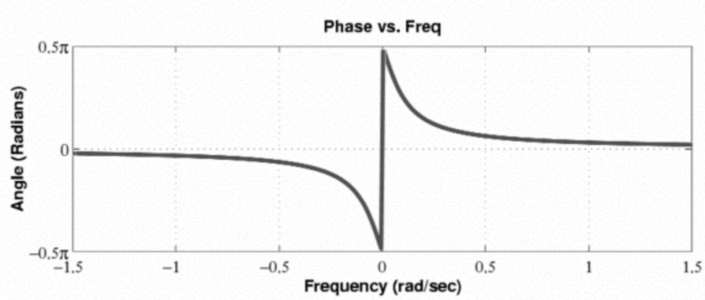
At $\omega = \infty$, $|H(j\infty)|^2 = \lim_{\omega\to\infty} \frac{\omega^2}{0.01+\omega^2} = \lim_{\omega\to\infty} \frac{\omega^2}{\omega^2} = 1$

At $\omega = 0.1$, $|H(j0.1)|^2 = \frac{0.01}{0.01+0.01} = \frac{1}{2}$

$$\angle H(j\omega) = \angle j\omega - \angle(0.1+j\omega) = \begin{cases} \pi/2 - \text{ArcTan}\left(\frac{\omega}{0.1}\right) & \text{if } \omega > 0 \\ -\pi/2 - \text{ArcTan}\left(\frac{\omega}{0.1}\right) & \text{if } \omega < 0 \end{cases}$$

Plots from MATLAB are below:





(c) From the plot in part (b), the max value is one as $\omega \to \infty$. Also $|H(j\omega)|^2 = 1/2$ at $\omega = 0.1$ rad/s.

why is it called "3dB point"?

$$10\log_{10}|H(j0.1)|^2 = 10\log_{10}\left(\tfrac{1}{2}\right) = 10(-0.301) = -3.01\text{ dB}$$

Notice that $10\log_{10}|H(j\infty)|^2 = 10\log_{10}(1) = 0$, so the decibel value at $\omega = 0.1$ rad/s is -3.01 dB down from the maximum dB value.

(d) Use <u>SUPERPOSITION</u> to do each input separately and then add them together.

$$x(t) = \underbrace{10}_{x_1(t)} + \underbrace{20\cos(0.1t)}_{x_2(t)} + \underbrace{\delta(t-0.2)}_{x_3(t)}$$

① $x_1(t)$ is a sinusoid whose frequency is zero. Thus we need $H(j\omega)$ at $\omega = 0$. $H(j0) = \dfrac{j0}{0.1+j0} = 0$

$\Rightarrow y_1(t) = 0$

② $x_2(t)$ is a sinusoid with $\omega = 0.1$ rad/s.

$H(j\omega)$ at $\omega = 0.1$ is $H(j0.1) = \dfrac{j0.1}{0.1+j0.1} = \dfrac{j}{1+j}$

We need $H(j0.1)$ in <u>POLAR</u> form.

$$H(j0.1) = \frac{j}{1+j} = \frac{j(1-j)}{(1+j)(1-j)} = \frac{j+1}{2} = \frac{\sqrt{2}}{2}e^{j\pi/4}$$

$$\Rightarrow y_2(t) = \left(\frac{\sqrt{2}}{2}\right)20\cos(0.1t+\pi/4) = 10\sqrt{2}\cos(0.1t+\pi/4)$$

③ for $x_3(t)$ we have a shifted impulse, so use $h(t)$.

$$y_3(t) = h(t-0.2) = \delta(t-0.2) - 0.1e^{-0.1(t-0.2)}u(t-0.2)$$

Now, add them together:

$$y(t) = y_1(t) + y_2(t) + y_3(t)$$

$$y(t) = 10\sqrt{2}\cos(0.1t+\pi/4) + \delta(t-0.2) - 0.1e^{-0.1(t-0.2)}u(t-0.2)$$

PROBLEM 10.5:

(a) The period is $T_0 = 8$, so $\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{8} = \frac{\pi}{4}$ rad/s

$$a_k = \frac{1}{8}\int_{-1}^{1} 10\, e^{-j\frac{\pi}{4}kt}\, dt$$

The limits on the integral are NOT -4 to $+4$ because $x(t)$ is ZERO for $-4 \le t < -1$ and $1 < t \le 4$.

(b) To plot the spectrum, we need the values of a_k for $k = -4, -3, -2, -1, 0, 1, 2, 3, 4$.

At $k=0$ use L'Hôpital's rule or take $\lim\limits_{k\to 0}$

$$a_0 = \frac{10\,(\pi k/4)}{\pi k} = \frac{10}{4} = 2.5$$

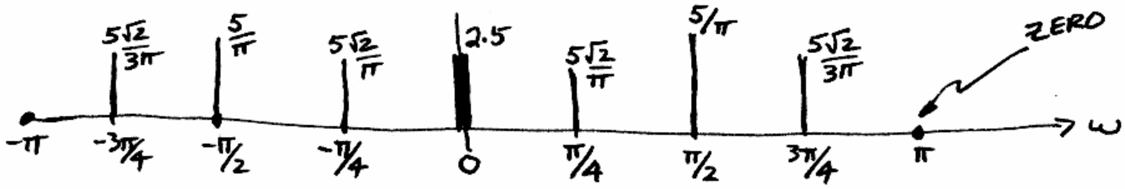
$$a_1 = \frac{10 \sin(\pi/4)}{\pi} = \frac{10 \cdot \sqrt{2}/2}{\pi} = \frac{5\sqrt{2}}{\pi}$$

NOTE: $a_{-1} = a_1$ and generally we have $a_k = a_{-k}$

$$a_2 = \frac{10 \sin(\pi/2)}{2\pi} = \frac{10}{2\pi} = \frac{5}{\pi} = a_{-2}$$

$$a_4 = \frac{10 \sin(\pi)}{\pi k} = 0$$

$$a_3 = \frac{10 \sin(3\pi/4)}{3\pi} = \frac{10\sqrt{2}/2}{3\pi} = \frac{5\sqrt{2}}{3\pi} = a_{-3}$$

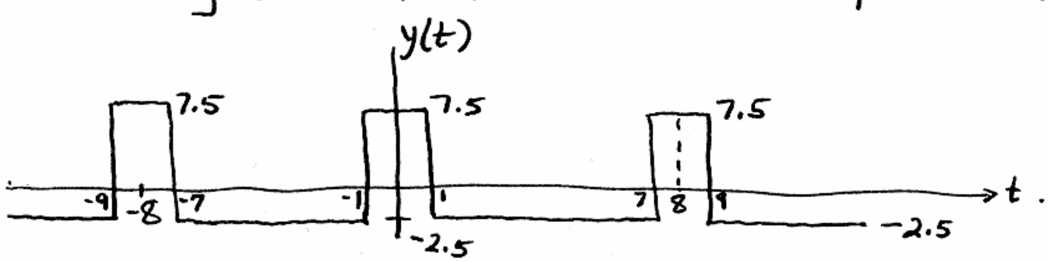


(c) The frequency response of the filter will MULTIPLY the spectrum of the input. Thus the spectrum of the output will be everything EXCEPT the line at DC.

Thus $y(t)$ has a FOURIER Series that is identical to the FS for $x(t)$ except the a_0 term is missing

$$\Rightarrow \quad y(t) = x(t) - a_0 = x(t) - 2.5$$

Subtracting a constant will shift the plot down



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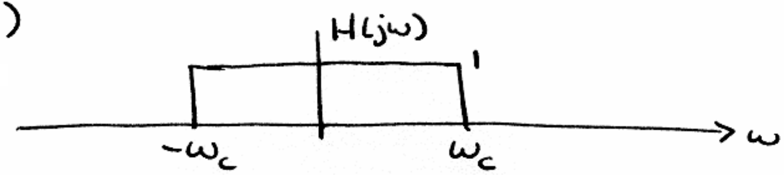
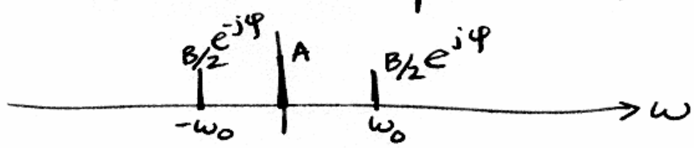
(d)

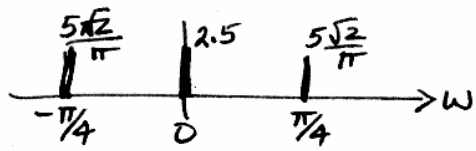


Again, we note that $H(j\omega)$ will MULTIPLY the spectrum of $x(t)$. We want the spectrum of the output to be



Since $\omega_0 = \pi/4$, we need $\omega_c > \omega_0$. But we also need $\omega_c < 2\omega_0$. Thus $\frac{\pi}{4} < \omega_c < \frac{\pi}{2}$

With this ω_c, the spectrum of $y(t)$ will be:



$\Rightarrow$

$$y(t) = 2.5 + \frac{10\sqrt{2}}{\pi}\cos\left(\frac{\pi}{4}t\right)$$

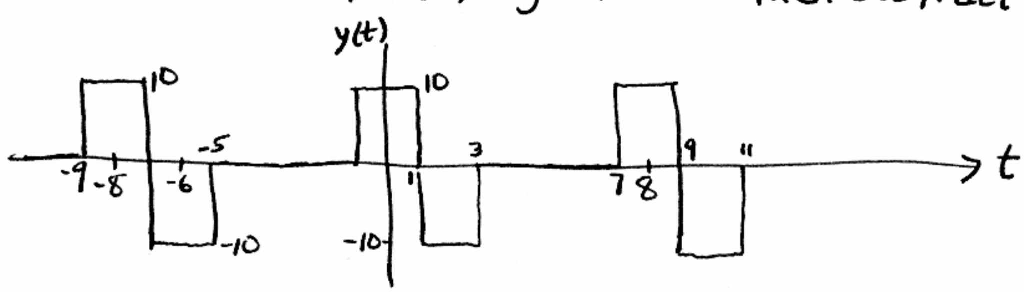
(e) If $H(j\omega) = 1 - e^{-j2\omega}$ we can find $h(t)$ by doing an INVERSE FOURIER TRANSFORM.

$$h(t) = \delta(t) - \delta(t-2)$$

Then $y(t) = x(t) * h(t) = x(t) * \delta(t) - x(t) * \delta(t-2)$

$$= x(t) - x(t-2).$$

So we must shift $x(t)$ by 2 and then subtract



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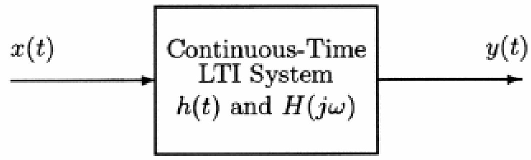
PROBLEM 10.6:



The periodic input to the above system is defined by the equation:

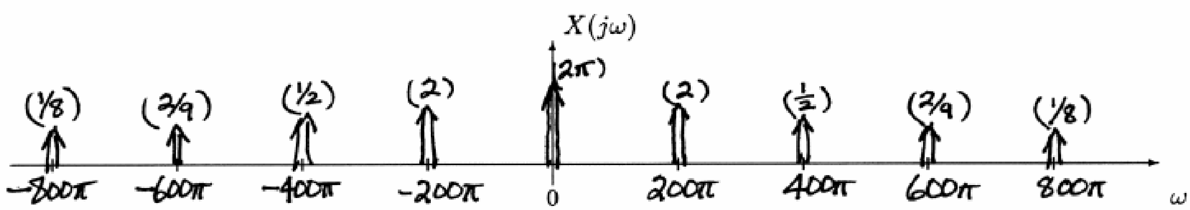
$$x(t) = \sum_{k=-4}^{4} a_k e^{j200\pi kt}, \quad \text{where} \quad a_k = \begin{cases} \dfrac{1}{\pi |k|^2} & k \neq 0 \\ 1 & k = 0 \end{cases}.$$

$a_1 = 1/\pi$
$a_2 = 1/4\pi$
$a_3 = 1/9\pi$
$a_4 = 1/16\pi$

(a) Determine the Fourier transform of the periodic signal $x(t)$. Give a formula and then plot it on the graph below. Label your plot carefully to receive full credit.

$$X(j\omega) = \sum_{k=-4}^{4} 2\pi a_k \delta(\omega - 200\pi k)$$

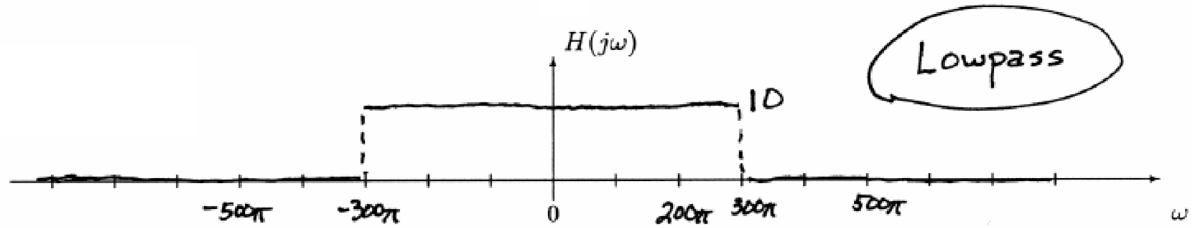
$\omega_0 = 200\pi$



(b) The frequency response of the LTI system is given by the following equation:

$$H(j\omega) = \begin{cases} 10 & |\omega| \leq 300\pi \\ 0 & |\omega| > 300\pi \end{cases}.$$

Plot this function on the graph below using the same frequency scale as the plot in part (a). *Note carefully what type of filter (i.e., lowpass, bandpass, highpass) this is.*



Lowpass

(c) Write an equation for $y(t)$. In order to get full credit you must simplify it to include only cosine functions.

$$Y(j\omega) = H(j\omega)X(j\omega) = 20\pi\delta(\omega) + 20\delta(\omega - 200\pi) + 20\delta(\omega + 200\pi)$$

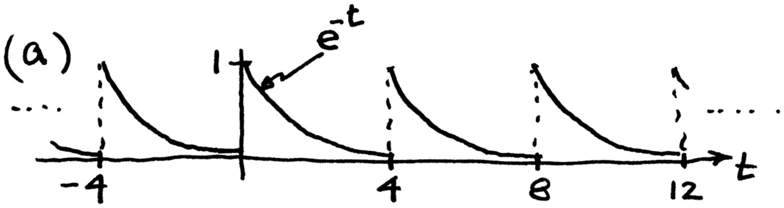
Invert: $y(t) = 10 + \underbrace{\frac{10}{\pi} e^{j200\pi t} + \frac{10}{\pi} e^{-j200\pi t}}_{\text{use Euler's inverse formula}}$

$$y(t) = 10 + \frac{20}{\pi}\cos(200\pi t)$$

PROBLEM 10.7:

$x(t) = e^{-t}$ for $0 \le t < 4$ repeats with a period of $T=4$

(a)



(b) $\omega_0 = 2\pi/T = 2\pi/4 = \pi/2$ rad/s

(c)
$$a_k = \frac{1}{4}\int_0^4 e^{-t} e^{-j\frac{\pi}{2}kt}\,dt = \frac{1}{4}\int_0^4 e^{-(t+j(\frac{\pi}{2})kt)}\,dt$$

$$a_k = \frac{1}{4}\left.\frac{e^{-(t+j(\frac{\pi}{2})kt)}}{-(1+j\pi k/2)}\right|_0^4$$

$$a_k = \frac{1}{4}\left(\frac{1-e^{-(1+j\pi k/2)4}}{(1+j\pi k/2)}\right)$$

general formula for all k

Evaluate a_k for $k = 0, 1, -1$

$a_0 = 0.2454$ $\quad a_1 = 0.1318 e^{-j0.32\pi}$ $\quad a_{-1} = a_1^*$

(d) If the cutoff frequency of the LPF is $\omega_{co} = 2\pi/3$, then only those terms whose frequency satisfies

$$|k\omega_0| < 2\pi/3 \iff |k\pi/2| < 2\pi/3 \Rightarrow |k| < 4/3$$

will pass through the filter.

Thus, the output is

$$y(t) = a_0 + a_1 e^{j\omega_0 t} + a_{-1} e^{-j\omega_0 t}$$

$$= 0.2454 + 0.1318\, e^{-j0.32\pi} e^{j\pi t/2} + 0.1318 e^{j0.32\pi} e^{j\pi t/2}$$

$$\Rightarrow y(t) = 0.2454 + 0.2636 \cos\left(\frac{\pi}{2}t - 0.32\pi\right)$$

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PROBLEM 10.8:

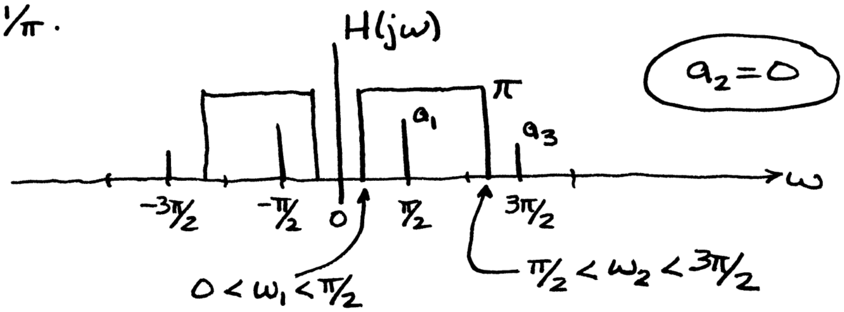
(a) $\omega_0 = 2\pi/T_0 = 2\pi/4 = \pi/2$ rad/s

(b) The Fourier Series coefficients for the 50% duty cycle square wave were derived in Chapter 3

$$a_k = \begin{cases} \frac{1}{2} & k=0 \\ 0 & k=\pm 2, \pm 4, \pm 6, \ldots \\ \frac{\sin(\pi k/2)}{\pi k} & k=\pm 1, \pm 3, \pm 5, \ldots \end{cases}$$

(c) $y(t) = 2\cos\left(\frac{2\pi t}{4}\right) = 2\cos\left(\frac{\pi}{2}t\right)$

Since the frequency of $y(t)$ is $\pi/2$ which is ω_0 the filter just needs to pass a_1 & a_{-1}. Also, the gain of the BPF needs to be π because $|a_1| = 1/\pi$.



$$H(j\omega) = \begin{cases} 0 & |\omega| < \omega_1 \\ \pi & \omega_1 \le |\omega| \le \omega_2 \\ 0 & \omega_2 \le |\omega| \end{cases}$$

(d) $y(t) = 2\cos\left(\frac{2\pi}{3}t\right)$

The frequency of $y(t)$ is $\frac{2\pi}{3}$ rad/s which is <u>NOT</u> an integer multiple of $\omega_0 = \frac{\pi}{2}$. Hence, there is no LTI system that will have $y(t)$ as its output when the square wave $x(t)$ is the input.

PROBLEM 10.9:

For each filter (1 through 7), determine the output and then do the matching.

1. $H(j\omega)$ is a highpass filter. All components except DC are passed, so the output is $y(t) = x(t) - a_0$
$$= x(t) - 1/2$$

2. $H(j\omega) = e^{-j\omega/2}$ corresponds to a pure delay of 1/2.
$$y(t) = x(t - 1/2)$$

3. Since the input signal only contains the discrete frequencies, $\omega_k = k\omega_0$, we evaluate $H(j\omega)$ at $\omega = k\omega_0$.
$$H(jk\omega_0) = \tfrac{1}{2}\left(1 + \cos(k\omega_0 T_0)\right) = \tfrac{1}{2}\left(1 + \cos(2\pi k)\right)$$
$$= \tfrac{1}{2}(1+1) = 1. \quad \Rightarrow \quad y(t) = x(t)$$

4. This LPF passes DC and the lines at $\omega = \pm\omega_0$
$$y(t) = a_0 + a_1 e^{j\omega_0 t} + a_{-1} e^{-j\omega_0 t}$$
$$= \tfrac{1}{2} + \tfrac{1}{\pi} e^{j\omega_0 t} + \tfrac{1}{\pi} e^{-j\omega_0 t} = \tfrac{1}{2} + \tfrac{2}{\pi}\cos(\omega_0 t)$$

5. This LPF passes DC only $\Rightarrow$ $y(t) = 1/2$

6. This LPF has a delay of 1/2 & passes $\omega = 0, \pm\omega_0$
$$\Rightarrow \quad y(t) = \tfrac{1}{2} + \tfrac{2}{\pi}\cos(\omega_0(t - 1/2))$$

7. This BPF passes only the lines at $\omega = \pm\omega_0$
$$\Rightarrow \quad y(t) = \tfrac{2}{\pi}\cos(\omega_0 t)$$

Now do the matching:

(a) 5 (c) 7 (e) 2

(b) 6 (d) 1